In this exam $\mathbb{R}$ denotes the field of all real numbers; $\mathbb{R}^d$ is the d-dimensional Euclidean space with the usual norm $\|x\| = \left(\sum_{k=1}^{d} x_k^2\right)^{1/2}$. Proofs or counterexamples are required for all problems.

1. Let f be continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose $f(0)=0$ and $|f'(x)| \le M$ for some $M>0$ and all $x \in (0,1)$.
   (a) Prove that $|f(x)| \le M$ for $x \in [0,1]$.
   (b) Prove that
   $$\left|\big(f(x)\big)^2 - \big(f(y)\big)^2\right| \le 2M^2|x-y|$$
   for all $x, y \in [0,1]$.

2. Let $f : [0,\infty) \to \mathbb{R}$ be a monotone increasing function.
   (a) Give a reasonable mathematical definition of what it means to say $\lim_{x\to\infty} f(x) = a$ with a a real number.
   (b) Prove that if the improper integral $\int_0^\infty f(x)\,dx$ exists, then $\lim_{x\to\infty} f(x) = 0$.
   (c) Is the converse of the statement in part (b) true? Prove or give a counterexample.

3. Let $\{x_n\}$ be a sequence in $\mathbb{R}^d$ such that for all $n \ge 1$,
   $$\|x_{n+1} - x_n\| \le \frac{1}{n^2}.$$
   Prove that the sequence $\{x_n\}$ is Cauchy.

4. Prove that the function $g(x) = \frac{1}{1+x}$ is uniformly continuous on $[0,\infty)$.

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear injective map. Show that there exists a constant $m>0$ such that $\|T(x)\| \ge m\|x\|$ for all $x \in \mathbb{R}^n$.

6. Let $v_1, \ldots, v_m$ be linearly independent elements in a vector space V, and let $w = \frac{1}{m}\left(v_1 + \ldots + v_m\right)$.
   (a) Show that the list of vectors $\{v_i - w\}_{i=2}^m$ is linearly independent.
   (b) Show that the list of vectors $\{v_i - w\}_{i=1}^m$ is not linearly independent.

7. Let V be a vector space and let T be a linear map on V. Suppose $\dim \operatorname{null}(T^2) = 5$. Prove that $\dim \operatorname{null}(T) \ge 3$.

8. Let V be a finite-dimensional inner product space of dimension n, with the inner product of vectors u and v in V denoted by $u \cdot v$. Let $\mathcal{B} = \{x_1, \ldots, x_n\}$ be a basis of V.
   (a) For each $i = 2, \ldots, n$ let
   $$x_i' = x_i - \frac{x_i \cdot x_1}{x_1 \cdot x_1}\, x_1$$
   Prove that $x_i' \cdot x_1 = 0$ and that $\mathcal{B}' := \{x_1, x_2', \ldots, x_n'\}$ is a basis of V.
   (b) Prove that V has a basis consisting of pairwise orthogonal vectors.