1. (25 points) Let a random variable $X$ have the binomial distribution $b(p,n)$, and let the function $g(p)$ defined as $g(p) = p(1 - p)$.

(a) Show that the UMVUE of $g(p)$ is $\hat{\delta} = X(n - X)/n(n - 1)$.

(UMVUE: uniformly minimum variance unbiased estimator).

(b) Determine the limiting distribution of $\sqrt{n}(\hat{\delta} - g(p))$ and $n(\hat{\delta} - g(p))$ when $g'(p) \neq 0$ and $g'(p) = 0$, respectively.

2. (25 points) Let the random variable $X$ follow the inverse Gaussian distribution $I(\mu, \tau)$ with density $\sqrt{\tau/2\pi x^3} \exp\left(-\frac{\tau}{2\mu^2}(x - \mu)^2\right)$, $x > 0$, $\tau, \mu > 0$.

(a) Find the moment generating function of $X$.

(b) Show that $V = \frac{\tau}{X^2\mu^3}(X - \mu)^2 \sim \chi^2_1$

Let $X_1, \ldots, X_n$ be a random sample from $I(\mu, \tau)$.

(c) Show that $\bar{X} = \sum_{i=1}^{n} X_i/n \sim I(n\tau)$.

(d) Show that there exists a UMP test for testing $H_0: \mu \leq \mu_0$ versus $H_1: \mu > \mu_0$ when $\tau$ is known. (UMP: uniformly most powerful).

3. (25 points) Let $X_1, X_2, \ldots, X_m$ be a random sample from an exponential distribution with mean $\lambda$, and $Y_1, Y_2, \ldots, Y_n$ be a random sample form an exponential distribution with mean $\mu$, and assume that the two samples are independent.

(a) Find the LRT statistic, $T$, for testing the null hypothesis $H_0: \lambda = \mu$ versus the alternative hypothesis $H_1: \lambda \neq \mu$. (LRT: Likelihood Ratio Test).

(b) Using a suitable one-to-one transformation of $T$, find the exact 5% critical region for the LRT in (a). Give the critical region in terms of the percentile(s) of a known distribution.

Clearly identify the distribution and which percentiles(s), upper or lower.

4. (25 points) Let $X_1, \ldots, X_n$ be a random sample from $U(\theta, \theta + 1)$, where $-\infty < \theta < \infty$ and it is unknown. Assume a prior distribution for $\theta$ given by the probability density function, for $-\infty < \theta < \infty$,

$$\pi(\theta) = \frac{1}{2} e^{-|\theta|} .$$

(a) Find the posterior distribution of $\theta$, given $(X_1 = x_1, \ldots, X_n = x_n)$, i.e., $\pi(\theta|x_1, \cdots, x_n)$.

(b) Find the Bayes estimator of $\theta$ under the loss function, $L(\theta, \delta) = (\theta - \delta)^2$. 