In this exam \( \mathbb{R} \) denotes the field of all real numbers; \( \mathbb{R}^d \) is the \( d \)-dimensional Euclidean space with the usual norm \( \|x\| = \left( \sum_{k=1}^{d} x_k^2 \right)^{1/2} \); \( C[0,1] \) is the space of continuous functions on the interval \([0,1]\). Proofs or counterexamples are required for all problems.

1. If \( f \) is continuous on \([a,b]\), if \( a < c < d < b \), and \( M = f(c) + f(d) \), prove that there exists a number \( \xi \) between \( a \) and \( b \) such that \( M = 2f(\xi) \).

2. Prove that if a set \( C \) in \( \mathbb{R}^d \) is connected and a point \( x \in \mathbb{R}^d \) is a cluster point of \( C \), then the set \( C \cup \{x\} \) is connected.

3. Prove the Monotone Convergence Theorem for Sequences as stated below. Note: For the “only if” part, do not simply state that a convergent sequence is bounded; prove it.

   Let \( \{x_n\} \) be a monotone increasing sequence of real numbers. Then \( \{x_n\} \) is convergent if and only if it is bounded.

4. Prove or give a counterexample: Let \( f \) and \( g \) be two functions on the interval \([-1,1]\). If the product \( fg \) is Riemann integrable on \([-1,1]\), then at least one of \( f \) and \( g \) must be Riemann integrable on \([-1,1]\). Carefully support all your statements.

5. Let \( X = \{ f \in C[0,1] : f(0) = 0 \} \). You may assume that \( X \) is a vector space over \( \mathbb{R} \). For each \( f \in X \), let \( (Tf)(x) = \int_0^x f(y) \, dy \), \( x \in [0,1] \).
   (a) Show that \( T \) is a linear map from \( X \) to itself.
   (b) Show that \( T \) is injective.
   (c) Show that \( T \) is not surjective.

6. Let \( V \) be a vector space, and \( T : V \to V \) a linear map. Suppose there exist linearly independent vectors \( v_1, v_2, v_3 \) such that \( Tv_1 = v_2, \; Tv_2 = v_3, \; \) and \( Tv_3 = v_2 \). Show that \( \lambda = 0, \; \lambda = 1, \) and \( \lambda = -1 \) are eigenvalues of \( T \). (Hint: consider appropriate linear combinations of \( v_1, v_2, \) and \( v_3 \) as possible eigenvectors.)

7. Let \( \ell^2 \) be the set of all real sequences \( \{a_n\}_1^\infty \) such that \( \sum_{n=1}^{\infty} |a_n|^2 < \infty \). Prove that \( \ell^2 \) is a vector space over \( \mathbb{R} \) and that \( \langle \{a_n\}, \{b_n\} \rangle := \sum_{n=1}^{\infty} a_n b_n \) defines an inner product on \( \ell^2 \).

8. Let \( V \) be a finite-dimensional vector space and \( T \) a linear map from \( V \) to itself. Suppose \( \text{range} (T - 2I) \subseteq \text{null} (T - 3I) \). Show that \( T \) is invertible.