UC Calculus Contest April 7, 2015

Name:____________________ **M#:**______________ **Instructor:**_________________

Instructions: *This exam has seven problems on seven pages. Show all your work, expressing yourself in clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.*

1

Let S_1 be the 1×1 square. Define by induction the squares S_{i+1}equal to the square obtained by connecting the midpoints of the sides of the square S_i. Find $\sum_{i=1}^{\infty}$perimeter (S_i) and $\sum_{i=1}^{\infty}$area (S_i).

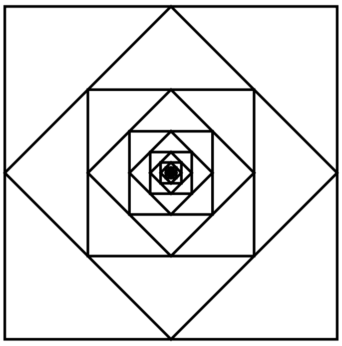

Figure 1.

1 solution

The largest square has the side-length equal to 1. The second largest square has the side length equal to $\sqrt{2}\ \frac{1}{2} = \frac{1}{\sqrt{2}}$, and so on. Therefore,

$$\sum_{i=1}^{\infty} \text{perimeter}(S_i) = 4\sum_{n=0}^{\infty}\frac{1}{\sqrt{2}^{n}} = 4\,\frac{1}{1-\frac{1}{\sqrt{2}}} = 4\left(2+\sqrt{2}\right)$$

and

$$\sum_{i=1}^{\infty} \text{area}(S_i) = \sum_{n=0}^{\infty}\frac{1}{\sqrt{2}^{2n}} = \sum_{n=0}^{\infty}\frac{1}{2^{n}} = 2$$

2

Suppose that a, b are nonzero real numbers and f is differentiable at x. Express the limit

$$\lim_{h\to 0} \frac{f(a\,h+x)-f(b\,h+x)}{h}$$

In terms of a, b and $f'(x)$.

2 solution

$$\lim_{h\to 0} \frac{f(a\,h+x)-f(b\,h+x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(f(a\,h+x)-f(x)-f(b\,h+x)+f(x)\right) =$$

$$\lim_{h\to 0} \frac{f(a\,h+x)-f(x)}{h} - \lim_{h\to 0} \frac{f(b\,h+x)-f(x)}{h} = a \lim_{h\to 0} \frac{f(a\,h+x)-f(x)}{a\,h} - b$$

$$\lim_{h\to 0} \frac{f(b\,h+x)-f(x)}{b\,h} = a\,f'(x) - b\,f'(x).$$

3

Consider the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

(a) Show that the series converges.

(b) Find the sum of the series.

3 solution

For $x \in (-2, 2)$,

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \frac{1}{1 - x/2} = \frac{2}{2 - x}$$

$$\frac{2}{(2 - x)^2} = \frac{d}{dx} \frac{2}{2 - x} = \sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{2^n}$$

$$\frac{2\, x}{(2 - x)^2} = \sum_{n=1}^{\infty} \frac{n\, x^n}{2^n}$$

$$\frac{2}{(2 - x)^2} + \frac{4\, x}{(2 - x)^3} = \frac{d}{dx} \frac{2\, x}{(2 - x)^2} = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{n\, x^n}{2^n} = \sum_{n=1}^{\infty} \frac{n^2\, x^{n-1}}{2^n}$$

Setting $x = 1$,

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6.$$

4

Let x be a real number. Show that the limit exists, and then find it:

$$\lim_{n\to\infty} \sin(\ ...\ \sin(\sin(\sin(\sin(x)))))$$

(above, the sin is calculated n times).

4 solution

Except for, possibly, the first two terms, the sequence is monotone (decreasing and positive if $\sin(x) > 0$, increasing and negative if $\sin(x) < 0$, constant if $\sin(x) = 0$). Hence, convergent, and

$$L = \lim_{n\to\infty} \sin(\ ...\ \sin(\sin(\sin(\sin(x)))))$$

Then $L \in (-1,\ 1)$, and

$$L = \sin(L)$$

and therefore

$$L = 0.$$

5

Assuming that $a > -1$ and $b > -1$, use Riemann sums to calculate

$$\lim_{n\to\infty} n^{b-a} \frac{1^a + 2^a + 3^a + \ldots + n^a}{1^b + 2^b + 3^b + \ldots + n^b}.$$

5 solution

$$\lim_{n\to\infty} n^{b-a} \frac{1^a + 2^a + 3^a + \ldots + n^a}{1^b + 2^b + 3^b + \ldots + n^b} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \left(\frac{3}{n}\right)^a + \ldots + \left(\frac{n}{n}\right)^a}{\left(\frac{1}{n}\right)^b + \left(\frac{2}{n}\right)^b + \left(\frac{3}{n}\right)^b + \ldots + \left(\frac{n}{n}\right)^b} =$$

$$\lim_{n\to\infty} \frac{\frac{1}{n}\left(\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \left(\frac{3}{n}\right)^a + \ldots + \left(\frac{n}{n}\right)^a\right)}{\frac{1}{n}\left(\left(\frac{1}{n}\right)^b + \left(\frac{2}{n}\right)^b + \left(\frac{3}{n}\right)^b + \ldots + \left(\frac{n}{n}\right)^b\right)} = \left(\lim_{n\to\infty} \frac{1}{n}\left(\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \left(\frac{3}{n}\right)^a + \ldots + \left(\frac{n}{n}\right)^a\right)\right) \Big/$$

$$\left(\lim_{n\to\infty} \frac{1}{n}\left(\left(\frac{1}{n}\right)^b + \left(\frac{2}{n}\right)^b + \left(\frac{3}{n}\right)^b + \ldots + \left(\frac{n}{n}\right)^b\right)\right) = \frac{\int_0^1 x^a \, dx}{\int_0^1 x^b \, dx} = \frac{1+b}{1+a}.$$

6

It is known (to be proved in the Multivariable Calculus) that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\,.$$

Show that

$$\int_1^{\infty} \left(\frac{1}{x}\right)^{\ln(x)} dx = \frac{\sqrt{\pi}}{2}\, e^{1/4} \left(\operatorname{erf}\left(\frac{1}{2}\right) + 1\right)$$

where

$$\operatorname{erf}(z) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt.$$

6 solution

$$\int_1^{\infty} \left(\frac{1}{x}\right)^{\ln(x)} dx = \int_0^{\infty} \left(\frac{1}{e^u}\right)^u e^u\, du = \int_0^{\infty} e^{-u^2}\, e^u\, du = \int_0^{\infty} e^{-u^2+u}\, du = \int_0^{\infty} e^{-\left(u-\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$du = \int_0^{\infty} e^{-\left(u-\frac{1}{2}\right)^2} e^{1/4}\, du = e^{1/4} \int_0^{\infty} e^{-\left(u-\frac{1}{2}\right)^2} du = e^{1/4} \int_{-1/2}^{\infty} e^{-u^2}\, du =$$

$$e^{1/4} \left(\frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{1}{2}\right) + \int_0^{\infty} e^{-u^2}\, du\right) = e^{1/4} \left(\frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{1}{2}\right) + \frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$e^{1/4} \left(\operatorname{erf}\left(\frac{1}{2}\right) + 1\right).$$

7

Show that series

(a)

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\ln(n)}$$

and

(b)

$$\sum_{n=1}^{\infty}\left(2^{1/n}-1\right)^{\ln(n)}$$

are both convergent.

7 solution

(a) One can argue

$$\sum_{n=8}^{\infty}\left(\frac{1}{n}\right)^{\ln(n)} \leq \sum_{n=8}^{\infty}\left(\frac{1}{n}\right)^{2} < \infty$$

or

$$\sum_{n=2}^{\infty}\left(\frac{1}{n}\right)^{\ln(n)} \leq \int_{1}^{\infty}\left(\frac{1}{x}\right)^{\ln(x)} dx = \frac{\sqrt{\pi}}{2}\, e^{1/4}\left(\operatorname{erf}\left(\frac{1}{2}\right)+1\right) < \infty.$$

(b) On the other hand,

$$2 < e = \lim_{n\to\infty}\left(\frac{1}{n}+1\right)^{n}.$$

Therefore, there exists n_0, such that for $n \geq n_0$

$$2 \leq \left(1+\frac{1}{n}\right)^{n}$$

$$2^{1/n} \leq 1+\frac{1}{n}$$

$$2^{1/n}-1 \leq \frac{1}{n}$$

$$\left(2^{1/n}-1\right)^{\ln(n)} \leq \left(\frac{1}{n}\right)^{\ln(n)}$$

$$\sum_{n=n_0}^{\infty} \left(2^{1/n} - 1\right)^{\ln(n)} \leq \sum_{n=n_0}^{\infty} \left(\frac{1}{n}\right)^{\ln(n)} < \infty.$$