Name:________________________ M#:__________ Instructor:____________________

Instructions: This exam has seven problems on seven pages. Show all your work, expressing yourself in clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.

1. Compute the limit or show that it does not exist.

$\lim_{x\to\infty} e^{\sin(2x)-x}$

Solution: We have $-1 \le \sin(2x) \le 1$ and so

$$e^{-1-x} \le e^{\sin(2x)-x} \le e^{1-x}.$$

Since $\lim_{x\to\infty} e^{-1-x} = \lim_{x\to\infty} e^{1-x} = 0$, we have $\lim_{x\to\infty} e^{\sin(2x)-x} = 0$, by the Squeeze Theorem.

2. Let $A = (-3, 7)$, $B = (1, 3)$, and $C = (5, 3)$ be three points in the plane. Sketch the triangle ABC and find the volume of the solid obtained by revolving the triangle around the line containing the altitude from the point A.

Solution: For convenience, we translate the triangle so that the point B is at the origin. The new points are then $A = (-4, 4)$, $B = (0, 0)$, and $C = (4, 0)$. The equation for the line AB is $y = -x$ or $x = -y$; that for the line AC is $y = -\frac{1}{2}x + 2$ or $x = 4 - 2y$.

We will use washers. For $0 \le y \le 8$, the area of the corresponding washer is $\pi((4 - 2y - (-4))^2 - (-y - (-4))^2)) = 3\pi(4 - y)^2$. The volume in question is then

$$V = 3\pi \int_0^4 (4 - y)^2\, dy = 64\pi.$$

3. Does the series $\displaystyle\sum_{n=1}^{\infty} n\, e^{-\sqrt{n}}$ converge or diverge? Fully justify your answer.

Solution:

Solution 1: We have $e^{\sqrt{n}} = 1 + \sqrt{n} + \frac{n}{2} + ... + \frac{n^2}{24} + ... + \frac{n^2\sqrt{n}}{125} + ...$ Hence, $e^{\sqrt{n}} \ge \frac{n^2\sqrt{n}}{125}$ and $n\, e^{-\sqrt{n}} \le \frac{125}{n^{3/2}}$. Since the series $\sum_{n=1}^{\infty} \frac{125}{n^{3/2}}$ converges as a p-series with $p = 3/2 > 1$, the original series converges by the direct comparison test.

Solution 2: Using the integral test, we consider the integral $\int_4^\infty xe^{-\sqrt{x}}\, dx$ (we chose 4 as the lower limit, because the function $xe^{-\sqrt{x}}$ is decreasing on the interval $[4, \infty)$ as can be seen from its derivative, $e^{-\sqrt{x}}(1-\sqrt{x}/2)$; any number greater than 4 would work as well). Setting $u=\sqrt{x}$, we get

$$\int_4^\infty xe^{-\sqrt{x}}\,dx = 2\int_2^\infty u^3e^{-u}\,du.$$

Integrating by parts three times, we see that this improper integral converges to $76e^{-2}$. Therefore, the original series converges.

4. Compute

$$\lim_{m\to\infty}\sum_{n=1}^{m}\frac{n^5}{m^6}$$

(Hint: Riemann sums.)

Solution: We write

$$\sum_{n=1}^{m}\frac{n^5}{m^6} = \frac{1}{m}\sum_{n=1}^{m}\left(\frac{n}{m}\right)^5$$

The expression on the right can be identified as the right-hand Riemann sum for the function $f(x)=x^5$ on the interval $[0,1]$. Indeed, take $m\geq 1$ and partition $[0,1]$ into m equal parts. Let $x_n=n/m$, for $k=0,\ldots,m$. The length of each subinterval is $\Delta x=1/m$. Then

$$R_m=\Delta x\sum_{n=1}^{m}f(x_n)=\frac{1}{m}\sum_{n=1}^{m}\left(\frac{n}{m}\right)^5.$$

Since f is continuous on $[0,1]$, we have

$$\lim_{m\to\infty}R_m=\int_0^1 f(x)\,dx=\int_0^1 x^5\,dx$$

or, written explicitly,

$$\lim_{m\to\infty}\sum_{n=1}^{m}\frac{n^5}{m^6}=\frac{1}{6}.$$

5. Let

$$f(x)=\begin{cases} xe^{-\frac{1}{x^2}}, & x\neq 0,\\ a, & x=0.\end{cases}$$

(a) Find a so that f is continuous for all x.

(b) Using your answer for part (a), show that f is differentiable at $x=0$ and find $f'(0)$.

Solution:

(a) The function is continuous for all $x \neq 0$ and $\lim_{x\to 0} xe^{-\frac{1}{x^2}} = 0$. Therefore, setting $a = 0$ makes f continuous on the whole line.

(b) From the definition of the derivative,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{xe^{-\frac{1}{x^2}} - 0}{x} = \lim_{x\to 0} e^{-\frac{1}{x^2}} = 0.$$

6. Compute

$$\int_0^\infty \frac{1}{1+x^3}\,dx$$

(Hint: use partial fractions; one factor of $x^3 + 1$ is $x + 1$.)

Solution: By partial fractions, we have

$$\frac{1}{1+x^3} = \frac{1}{3}\left[\frac{1}{x+1} + \frac{-x+2}{x^2-x+1}\right].$$

Completing the square in the second fraction gives

$$\frac{-x+2}{x^2-x+1} = -\frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}},$$

and so

$$\int \frac{1}{1+x^3}\,dx = \frac{1}{3}\left[\ln(x+1) - \frac{1}{2}\ln\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right) + \sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\right] + C.$$

Therefore,

$$\int_0^\infty \frac{1}{1+x^3}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{3}\left[\ln\frac{b+1}{\sqrt{(b-1/2)^2+3/4}} + \sqrt{3}\arctan\left(\frac{2b-1}{\sqrt{3}}\right)\right] - \frac{1}{3}\arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{3}\pi}{9}.$$

7. A function f on the real line is called mid-point convex if for any real numbers x_1 and x_2 one has

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

Show that if f is twice continuously differentiable and $f''(x) \geq 0$ for all x, then f is mid-point convex.

(Hint: what happens if you approximate $f(x_1)$ and $f(x_2)$ by the corresponding first degree Taylor polynomials centered at the point $x_0 = \frac{x_1+x_2}{2}$?)

Solution: Let $x_0 = \frac{x_1+x_2}{2}$. We have $f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + R$, where the error R is given by $R = \frac{1}{2}f''(\xi)(x_1 - x_0)^2$ for some point ξ between x_0 and x_1. Since $f'' \geq 0$, we have $f(x_1) \geq f(x_0) + f'(x_0)(x_1 - x_0)$ and similarly for $f(x_2)$. Therefore,

$$f(x_1) + f(x_2) \geq 2f(x_0) + f'(x_0)(x_1 + x_2 - 2x_0) = 2f(x_0),$$

and so f is mid-point convex.