# Statistics Prelim Exam Part 2 - Statistical Methods

10:00 am - 12:30 pm, Wednesday, May 7, 2025

1. Suppose $X_1, \ldots, X_n$ is a random sample from $N(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_m$ is an independent random sample from a $N(\mu_2, \tau^2)$ population, where $\mu_1, \mu_2$, $\sigma^2$ and $\tau^2$ are all unknown and $n, m \geq 2$.

   (a) Assume $\sigma^2 = \tau^2$. Suppose we wish to test $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$. Derive the likelihood ratio test (LRT) for this hypothesis (simplest possible form), and find an explicit expression for the exact rejection region of a size $\alpha$ test.

   (b) Assume $\mu_1 = \mu_2 = \mu$ and $\sigma^2 = \lambda\tau^2$ with $\lambda > 0$ and $\lambda$ known. Find the minimum sufficient statistics for the unknown parameters $\mu$ and $\tau^2$.

2. Let $X_1, \ldots, X_n$ be a random sample from a normal population as $\mathcal{N}(\theta, \sigma^2)$, and $\sigma^2 > 0$ is known. Suppose that the hypotheses to be tested are

$$H_0 : \theta \leq 0 \text{ versus } H_1 : \theta > 0.$$

   (a) Derive a uniformly most powerful (UMP) test of size $\alpha$ for testing the above hypotheses.

   (b) Assume a prior distribution on $\theta$ is $\mathcal{N}(0, \tau^2)$, where $\tau^2$ is known. Calculate the posterior probability that $H_0$ is true, i.e. $P(\theta \leq 0 | x_1, \ldots, x_n)$.

   (c) Show that $\lim_{\tau^2 \to \infty} P(\theta \leq 0 | x_1, \ldots, x_n) = p$-value for testing the hypotheses in Part (a).

   (d) For the special case $\sigma^2 = \tau^2 = 1$ and for sample mean values $\overline{x} > 0$, compare the values of $P(\theta \leq 0 | x_1, \ldots, x_n)$ and the p-value of the test derived in Part (a). Show that $P(\theta \leq 0 | x_1, \ldots, x_n)$ is always greater than the p-value.

3. Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution $E(a, b)$ with parameters $b > 0$ and $a \in (-\infty, +\infty)$ with the probability density function as

$$f(x) = \frac{1}{b} \exp\left[-\frac{(x-a)}{b}\right], \quad x \geq a.$$

   (a) When $b$ is known, show the distribution of $n[X_{(1)} - a]/b$ is $E(0, 1)$ and confirm that the uniformly minimum variance unbiased estimator (UMVUE) of $a$ is $X_{(1)} - (b/n)$, where $X_{(1)}$ is the miminum order statistic.

   (b) Assume that $b$ is known. Find the UMVUE of $P(X_1 \geq t)$ and $\frac{d}{dt}P(X_1 \geq t)$ for a fixed $t > 0$, where $X_1$ is arbitrarily the first random variable in the random sample.

   (c) Now assume $a$ is known and $b$ is unknown. Find the maximum likelihood estimator $\hat{g}(b)$ of $g(b) = P(X_1 \leq 2a)$. Clearly identify the asymptotic distribution of $\sqrt{n}(\hat{g}(b) - g(b))$ as $n \to \infty$.

4. Let $X_1, \ldots, X_n$ be a random sample from Poisson($\theta$) distribution, where $\theta$ is the mean parameter, $\theta > 0$ and $n \geq 1$.

   (a) Does an unbiased estimator of $1/\theta$ exist? Clearly justify your answer.

   (b) Find the uniformly minimum variance unbiased estimator (UMVUE) for $\tau(\theta) = \exp(-t\theta)$ with a fixed $t > 0$.