Statistics Prelim Exam Part 2 - Statistical Methods

10:00 am - 12:30 pm, Wednesday, August 20, 2025

1. Suppose $X_1, \ldots, X_n$ is a random sample from Uniform$(\theta, \theta+1)$, where $\theta \in (-\infty, \infty)$ and it is unknown. Assume a prior distribution for θ given by the probability density function as,

$$\pi(\theta) = \frac{1}{2}\exp(-|\theta|), \quad \theta \in (-\infty, \infty).$$

   (a) Find the posterior distribution of θ, given $(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$, i.e., $\pi(\theta|x_1, x_2, \ldots, x_n)$.

   (b) Find the Bayes estimator δ of θ under the loss function as $L(\theta, \delta) = (\theta - \delta)^2$.

2. Let $X_1, \ldots, X_n$ be a random sample from the inverse Gaussian distribution (InvGaussian(μ, λ)) with the probability density function as

$$f(x|\mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}}\exp\left[-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right], \quad x > 0, \;\; \mu, \lambda > 0.$$

   (a) Show that this density constitutes an exponential family.

   (b) Find the moment generating function $m(t)$ of $\bar{X}$, where $\bar{X} = \sum_{i=1}^{n} X_i/n$. Clearly specify its distribution with the related parameters.

   (c) Show that there exists a uniformly most powerful (UMP) test for testing $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$ when λ is known. Clearly derive the test giving the test statistic and the exact rejection region for a size α test.

3. Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution $\mathcal{E}(a, b)$ with parameters $b > 0$ and $a \in (-\infty, +\infty)$ with the probability density function as

$$f(x) = \frac{1}{b}\exp\left[-\frac{(x-a)}{b}\right], \quad x \geq a.$$

   (a) Assume that b is known. Show the distribution of $n[X_{(1)} - a]/b$ is $\mathcal{E}(0, 1)$ and confirm that the uniformly minimum variance unbiased estimator (UMVUE) of a is $X_{(1)} - (b/n)$, where $X_{(1)}$ is the miminum order statistic.

   (b) Assume that b is known. Find the UMVUE of $P(X_1 \geq t)$ and $\frac{d}{dt}P(X_1 \geq t)$ for a fixed $t > 0$, where X_1 is arbitrarily the first random variable in the random sample.

   (c) Assume a is known and b is unknown. Find the maximum likelihood estimator $g(\hat{b})$ of $g(b) = P(X_1 \leq 2a)$. Clearly identify the asymptotic distribution of $\sqrt{n}(g(\hat{b}) - g(b))$ as $n \to \infty$.

   (d) Assume both b and a are unknown. Find a likelihood ratio test of size α for testing $H_0 : b = b_0$ versus $H_1 : b \neq b_0$, giving the test statistic and the exact rejection region .

   Hint: For Part (d), you can use the following result without proving it:

   When both b and a are unknown, $X_{(1)}$ and $\sum[X_i - X_{(1)}]$ are jointly sufficient and complete. They are independently distributed as

$$\begin{aligned} n[X_{(1)} - a]/b &\sim \mathcal{E}(0, 1), \\ 2\sum_{i=1}^{n}[X_i - X_{(1)}]/b &\sim \chi^2_{2(n-1)}. \end{aligned}$$