Statistical Methods Prelim Exam

1:00 pm - 3:30 pm, Monday, August 15, 2022

UMVUE: uniformly minimum variance unbiased estimator

UMP: uniformly most powerful

i.i.d.: identically and independently distributed

1. Let $X_1, X_2, \cdots, X_n$ $(n \geq 5)$ be i.i.d. having the common probability density function

$$\sigma^{-1} \exp\{-(x-\mu)/\sigma\} I(x > \mu),$$

where μ is an unknown parameter, $-\infty < \mu < \infty$, but σ is known with $0 < \sigma < \infty$. The indicator function $I(\cdot) = 1$ if $x > \mu$ and 0 otherwise. Let $U = X_{(1)}$ be the smallest order statistic.

   (a) Show that $U = X_{(1)}$ is the complete sufficient statistic for μ.

   (b) Define $\overline{X} = \sum_{i=1}^{n} X_i/n$. Find the conditional expectation of $\overline{X}$, given that $U = u$, i.e., $E[\ \overline{X}\ |U = u]$.

   (c) Find the conditional expectation of $X_1 + e^{|X_1 - X_3|} - e^{|X_2 - X_4|}$, given that $U = u$, i.e., $E[\ X_1 + e^{|X_1 - X_3|} - e^{|X_2 - X_4|}\ \ |U = u]$.

2. Let $X_1, \ldots, X_n$ be an i.i.d. random sample from the normal population as $\mathcal{N}(\mu, \sigma^2)$.

   (a) Suppose that the hypotheses to be tested are

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0,$$

   where μ_0 is a known constant and assume $\sigma^2 > 0$ is known. Does there exist a UMP test of size α for the above hypotheses? If yes, derive the UMP test. If no, clearly justify your answer.

   (b) Now let the estimand be μ^2. Show that when σ is unknown, $\delta_n = \bar{X}^2 - \frac{S^2}{n(n-1)}$ is the UMVUE, where $\overline{X} = \sum_{i=1}^{n} X_i/n$ and $S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2$.

   (c) Derive the limiting distribution of $[\delta_n - \mu^2]$ as $n \to \infty$.

3. A Bayes action $\hat{\theta}(\mathbf{x})$ of a parameter θ based on a sample $\mathbf{x}$ is defined as an estimator $\delta(\mathbf{x})$ that minimizes the posterior expected loss, $\rho(\pi, \hat{\theta}(\mathbf{x})) = \int_{\Theta} L(\hat{\theta}(\mathbf{x}), \theta)\pi(\theta|\mathbf{x})d\theta$.

   (a) Prove that the posterior mean of θ is the Bayes action under the squared error loss, $L(\hat{\theta}(\mathbf{x}), \theta) = (\hat{\theta}(\mathbf{x}) - \theta)^2$.

   (b) Suppose $X_i \overset{i.i.d}{\sim} \mathcal{N}(\mu, \sigma^2), i = 1, \ldots, n$. The mean μ is assumed to be known and let the prior distribution of σ^2 be the inverse gamma distribution,

$$f(\sigma^2|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}}$$

   with $\alpha > 1$ and $\beta > 0$.

   Find the Bayes action of σ^2 under the squared error loss.

4. Let $X_{i1}, \ldots, X_{in_i}$ be two independent samples i.i.d. from the uniform distribution $\mathcal{U}(0, \theta_i)$, $i = 1, 2$, respectively, where $\theta_1 > 0$ and $\theta_2 > 0$ are unknown. Find a likelihood ratio test of size α for testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$. (Hint: find out the distribution of the likelihood ratio Λ by calculating the probability $P(\Lambda < c)$, where c is a constant).