PhD Preliminary Exam in Algebra - January 2021

Full marks may be obtained by complete answers to 4 questions.
Time allowed - 2 1/2 hours
No calculators, cell phones or other electronic devices allowed

(1) Let R be a principal ideal domain. Prove the following assertions.
  (a) Every non-zero prime ideal of R is maximal.
  (b) If S is an integral domain and $\phi : R \to S$ is a surjective ring homomorphism, then either ϕ is an ismorphism or S is a field.
  (c) If $R[x]$ is a prinicpal ideal domain, then R is a field.

(2) (a) Prove that $x^N + 1$ is irreducible in $\mathbb{Z}[x]$ if and only N is a power of 2.
  (b) Let $N = 2^n$, let q be a prime such that $q \equiv 1 \pmod{2N}$. Prove that $x^N + 1$ splits completely in F_q.

(3) Let $f(x) = x^3 - 11 \in \mathbb{Q}[x]$
  (a) Describe the splitting field E of $f(x)$ over $\mathbb{Q}$
  (b) Show that $[E : \mathbb{Q}] = 6$ and that the Galois group G is isomorphic to S_3.
  (c) Describe all the subgroups of G and the corresponding intermediate fields.

(4) Give examples (with appropriate justification) of finite extensions of fields $F \subset E$ where:
  (a) The extension is normal but not separable
  (b) The extension is separable but not normal
  (c) There are infinitely many intermediate fields K with $F \subset K \subset E$.

(5) Let ζ be a primitive 5-th root of unity.
  (a) How many intermediate fields are there between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$?
  (b) Let $u = \zeta + 1/\zeta$. Show that u is a root of a quadratic equation and that $\zeta^2 - u\zeta + 1 = 0$. How does this relate to your answer to (a)?
  (c) Deduce that the fifth roots of unity are of the form

$$\zeta = \frac{-1 + \epsilon\sqrt{5} \pm \sqrt{-10 - 2\epsilon\sqrt{5}}}{4}$$

  where $\epsilon = \pm 1$.