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MATHEMATICS QUALIFYING EXAM, AUGUST 17, 2021

Four Hour Time Limit

In this exam $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space.
Proofs, or counter examples, are required for all problems.

1. Fix $a < b$ and consider a function $f : [a, b] \to [a, b]$. Assume that there is a constant k such that $0 < k < 1$ and $|f(x) - f(y)| \le k\,|x - y|$ for all $x, y \in [a, b]$.
(a) Show that for each $x_0 \in [a, b]$, the sequence $(x_0, f(x_0), f(f(x_0)), \cdots)$ is a convergent sequence.
(b) Show that the limit of the above sequence is a fixed point of f.
(c) Show that f does not have any other fixed points in $[a, b]$.

2. Use the definitions to prove a theorem that if a sequence $\{f_n\}$ of real-valued continuous functions on $\mathbb{R}^d$ converges uniformly to f, then f is continuous.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(x) \to a \in \mathbb{R}$ and $f'(x) \to b \in \mathbb{R}$ as $x \to \infty$. Show that $b = 0$.

4. Show there exist two divergent series $\sum a_n$ and $\sum b_n$ of strictly positive terms such that if $c_n = \min\{a_n, b_n\}$, then $\sum c_n$ converges. You may use examples of convergent or divergent series from undergraduate calculus course without proof.

5. Consider a real vector space $Bil(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ of bilinear forms $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. For $i, j = 1, 2, \ldots, n$, let f_{ij} be a bilinear form given by $f_{ij}(u, v) = \langle u, e_i\rangle\langle v, e_j\rangle$. *(Here $u, v \in \mathbb{R}^n$, the inner product is the standard dot-product, and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. You do not need to verify that f_{ij} is bilinear.)*
(a) Prove that the set of n^2 bilinear forms $\{f_{ij} : i, j = 1, \ldots, n\}$ is linearly independent in $Bil(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$.
(b) Prove that every bilinear form $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a linear combination of $\{f_{ij} : i, j = 1, \ldots, n\}$.

6. (a) Find the matrix of the linear map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates a given vector by θ radians counterclockwise, with respect to the standard basis.
(b) Let the hyperbola H be given by the equation $xy = 1/4$. Find the equation of the hyperbola that is obtained by rotating H by $\pi/6$ radians counterclockwise.

7. Suppose that V is a real vector space with the inner product $\langle\cdot,\cdot\rangle$. If $T : V \to V$ is a linear transformation with adjoint T^* and $W \subset V$ is a T-invariant subspace, prove that the orthogonal complement $W^\perp$ is T^*-invariant.

8. Let $\mathcal{U} \subset \mathbb{R}^p$ be open, $f\colon \mathcal{U} \to \mathbb{R}^p$ be differentiable at point $\mathbf{c} \in \mathcal{U}$. Define $g\colon \mathcal{U} \to \mathbb{R}$ by $g(\mathbf{x}) = f(\mathbf{x}) \cdot \mathbf{x}$ for all $\mathbf{x} \in \mathcal{U}$. Show that g is differentiable at $\mathbf{c}$ and

$$Dg(\mathbf{c})(\mathbf{u}) = (Df(\mathbf{c})(\mathbf{u})) \cdot \mathbf{c} + f(\mathbf{c}) \cdot \mathbf{u} \qquad \text{for all } \mathbf{u} \in \mathbb{R}^p.$$

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