# Complex Analysis Prelim Exam
## UC Department of Math
## Jan 2020

(1) Suppose $(f_n)$ is a sequence of entire functions that converges locally uniformly in $\mathbb{C}$ to a polynomial $p$ with $m = \deg p > 0$. Prove that for all sufficiently large $n$, the number of zeroes of $f_n$ (counted according to multiplicity) is at least $m$.

(2) Suppose $f$ is a meromorphic function (recall this is a function that is holomorphic except on a set of isolated points that are poles) on $\mathbb{C}$ with $\lim_{z\to\infty} f(z) = 0$.

(a) Show that $f$ has finitely many poles in $\mathbb{C}$.

(b) Use part (a) to show that $f$ is a rational function. *Hint. Make adaptations to $f$ to turn it into entire function $g$. Think carefully about the growth of $g$ at $\infty$? What does this say about $g$?*

(3) Let $T$ be the Möbius transformation that maps $i, -i, \infty$ to $\omega, \bar{\omega}, 1$ respectively, where $\omega := e^{2\pi i/3}$. Determine the following images:
(a) $T(\{iy \mid y \in \mathbb{R}\})$,
(b) $T(\mathbb{H})$ where $\mathbb{H} = \{z \in \mathbb{C} \mid \mathfrak{Re}(z) > 0\}$ ,
(c) $T(\mathbb{D})$ where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

(4) Let $\Gamma$ be a piecewise smooth closed curve in $\mathbb{C} \setminus \mathbb{Z}$. Calculate

$$\int_\Gamma \frac{dz}{z(z^2-1)}.$$

*Hint: there are different cases to consider depending on what the curve $\Gamma$ does.*