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QUALIFYING EXAMINATION, AUGUST 18, 2020

In this exam $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space. Proofs, or counter examples, are required for all problems.

1. Let $E \subset \mathbb{R}$ be bounded and $f\colon E \to \mathbb{R}$ be uniformly continuous.
  (a) Prove that f is bounded on E.
  (b) Show that if f is assumed only to be continuous (instead of uniformly continuous), then it no longer follows that f is bounded.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Suppose $f(-1) = 0$, $f(1) = 0$, and that for all $x \in [-1, 1]$, we have $f''(x) \geq 0$. Prove that for all $x \in [-1, 1]$, $f(x) \leq 0$.

3. For continuous function $f : [0, 1] \to [0, 1]$, must there exist x in that interval for which $f(x) = x$?

4. (a) For a function $f : [0, \infty) \to \mathbb{R}$, define what it means to say that $\lim\limits_{x\to\infty} f(x) = a$ for some real a.
  (b) Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous with $f(0) = 0$ and $\lim\limits_{x\to\infty} f(x) = 1$. Prove that there exists a $c \in [0, \infty)$ such that $f(c) = \min_{x\in[0,\infty)} f(x)$.

5. (a) Define what it means to say that vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly independent.
  (b) Let A be an $n \times n$ matrix with real entries. If $v_1, \ldots, v_k \in \mathbb{R}^n$ are eigenvectors of A with distinct real eigenvalues, use the definition to show that $v_1, \ldots, v_k$ are linearly independent.
  Hint: Use mathematical induction.

6. Let V be a finite-dimensional vector space over $\mathbb{R}$, equipped with an inner product $\langle \cdot, \cdot \rangle$. Show that the orthogonal complement of the span of a finite nonempty collection of vectors $\{v_1, \ldots, v_k\} \subseteq V$ is equal to the intersection of the subspaces that are orthogonal complements to the span of $\{v_j\}$, $j = 1, \ldots, k$.

7. Let P_2 be the collection of all polynomials in x with coefficients in $\mathbb{R}$ with degree at most 2, and let $T : P_2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix},$$

where p' is the derivative of polynomials $p \in P_2$. Prove that T is onto but not one-to-one. Determine the kernel and the image of T. (You do not need to prove that T is a linear map.)

8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

  (a) Show that f is continuous at $(0, 0)$.
  (b) Compute $D_{(a,b)}f(0, 0)$ for any $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Here $D_{(a,b)}f(0, 0)$ denotes the directional derivative of f at the point $(0, 0)$ in the direction (a, b).
  (c) Is f differentiable at $(0, 0)$?

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