Statistical Methods Prelim Exam

9:30 am -12:00 pm, Thursday, August 22, 2019

1. Let $X_1, ..., X_n$ be i.i.d. with finite mean μ and variance $0 < \sigma^2 < \infty$. Let $S^2 = (1/(n-1)) \sum (X_i - \bar{x})^2$ where $\bar{X} = \sum X_i/n$.

   Show that S^2 is asymptotically normally distributed in the sense that $\sqrt{n}(S^2 - \sigma^2)$ converges in distribution to $N(0, c)$ for some constant $c > 0$. You may use the identity, $\sum (X_i - \mu)^2 = \sum (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$.

2. Let X_1, X_2, X_3 be i.i.d. $N(\mu, \sigma^2)$ with $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$. Assume that μ is unknown but σ is known. Let $T_1 = (X_1 + 2X_2 + 3X_3)/6$, $T_2 = X_1^2 - X_2^2 + \frac{1}{2}X_1 + |X_1X_3| - |X_2X_3|$ and $\bar{X} = (X_1 + X_2 + X_3)/3$.

   (a) Find the conditional distribution of X_1 given T_1.

   (b) Determine whether T_1 is sufficient for μ, and justify your answer.

   (c) Evaluate $E(T_2|\bar{X} = \bar{x})$.

3. Suppose that $X_1, \dots, X_m$ are i.i.d. Uniform$(-\theta, \theta)$, and $Y_1, \dots, Y_n$ are i.i.d. Uniform$(-2\theta, 2\theta)$ where $\theta \in (0, \infty)$ is assumed unknown. Assume that the X's and Y's are independent.

   (a) Find the most powerful level α test for $H_0 : \theta = 3$ versus $H_1 : \theta = 4$. Give the exact rejection region of the test. (Hint: You may first find the distribution $Y_1/2$).

   (b) Is this a uniformly most powerful test for $H_0 : \theta = 3$ versus $H_1 : \theta > 3$? Justify your answer.

4. Let $X_1, \dots, X_n$ be i.i.d. random sample from $U(\theta, \theta + 1)$, where $-\infty < \theta < \infty$ and it is unknown. Assume a prior distribution for θ given by the probability density function, for $-\infty < \theta < \infty$,

$$\pi(\theta) = \frac{1}{2}e^{-|\theta|} \ .$$

   (a) Find the posterior distribution of θ, and give an exact closed form expression for the posterior density function.

   (b) Find the Bayes estimate of θ with respect to the squared error loss function.