PhD Preliminary Exam in Algebra - Fall 2019

Full marks may be obtained by complete answers to 4 questions.
Time allowed - 2 1/2 hours
No calculators

(1) Say (giving reasons) which of the following field extensions $L \supset K$ are normal.
   (a) $K = \mathbb{Q}$, $L = \mathbb{Q}(\zeta)$, where ζ is a primitive p-root of unity and p is prime.
   (b) $K = \mathbb{Q}(\sqrt{-3})$, $L = K(\sqrt[3]{5})$
   (c) $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt[3]{3})$

   In each case find the order of the Galois group and find the fixed field of the Galois group. State clearly any result which you use.

(2) Show that an irreducible polynomial over a field of characteristic zero has distinct roots, and give an example of an irreducible polynomial over a field of characteristic p with a repeated root.

   Let F be a field with 3 elements and let f be the polynomial $X^3+X^2-1 \in F[X]$. Show that f is irreducible over F. If α is a root of f in a splitting field, show that $\alpha^2(1+\alpha)^2 = 1+\alpha$, and use this to express the remaining roots in terms of α. Deduce that $F(\alpha)$ is a splitting field of f over F.

(3) Let K be a field of characteristic zero such that every proper finite extension of K has even degree. Show that every such extension has degree a power of 2.

   Show that any proper finite extension of the real field $\mathbb{R}$ has even degree and deduce that the complex field $\mathbb{C}$ is algebraically closed.

(4) Suppose that $K \subset L$ are fields. Define what is meant by the degree $[L : K]$ of the extension $K \subset L$. Show that if $\alpha \in L$ then $[K(\alpha) : K]$ is equal to the degree of the minimal polynomial of α over K.

   Suppose that $K \subset L \subset M$ are fields. Show that if $[L : K]$ and $[M : L]$ are finite, then $[M : K] = [M : L].[L : K]$.

(5) Describe the Galois group of the polynomials
   a) $x^4 + 4x^2 - 5$
   b) $x^4 + 2$

   over $\mathbb{Q}$.