Ordinary Differential Equations Preliminary Exam
May 2019

1. Suppose that A is an $n \times n$ matrix that satisfies $A^2 = -A$.
   (a) Find an explicit form for e^{At} in terms of A, but without any series involving A.
   (b) Determine the stability of the origin of the linear system $\dot{x} = Ax$.

2. Consider the following linear system of equations:

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(\mathbf{0}) = \mathbf{x_0}.$$

   (a) Solve the linear system.
   (b) Find the stable, unstable, and center subspaces E^s, E^u, and E^c.
   (c) Sketch the phase portrait.

3. Consider the following system:

$$\begin{aligned} \dot{x} &= -y + x(\mu - x^2 - y^2), \\ \dot{y} &= x + y(\mu - x^2 - y^2). \end{aligned}$$

   Determine the equilibria and their stability. Draw the bifurcation diagram.
   (Hint: rewrite the system using polar coordinates.)

4. Let $V(x, y) = x^2(x-1)^2 + y^2$. Consider the dynamical system

$$\frac{dx}{dt} = -\frac{\partial V}{\partial x}, \qquad \frac{dy}{dt} = -\frac{\partial V}{\partial y}.$$

   (a) Find the critical points of this system and determine their linear stability.
   (b) Show that V decreases along any solution of the system.
   (c) Use (b) to prove that if $z_0 = (x_0, y_0)$ is an isolated minimum of V then z_0 is an asymptotically stable equilibrium.