PRELIMINARY EXAM

PARTIAL DIFFERENTIAL EQUATIONS MAY 2, 2023

FULL NAME:

ID NUMBER:

Instruction: Choose **only five** (out of six) problems to do. Each problem is worth 20 points.

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	20	20	100
Score:							

1. [20 points] Let $U \subset \mathbb{R}^n$, $n \geq 2$, be open, bounded, and connected with C^1 boundary. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a nonlinear function. Consider the boundary-value problem

$$\begin{cases} \Delta u = F(u) & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Assume that $u \mapsto \dfrac{F(u)}{u}$ is strictly increasing for all $u > 0$. Show that it is impossible to have two solutions u, v of the boundary-value problem above satisfying $u(x) > v(x) > 0$ for all $x \in U$.

2. [20 points] Let $U \subset \mathbb{R}^n$, $n \geq 2$, be open, bounded, and connected with C^1 boundary. Consider the boundary-value problem

$$\begin{cases} \Delta u = a(x)u + f(x) & \text{in } U, \\ u = h(x) & \text{on } \partial U. \end{cases}$$

The functions $a(x)$, $f(x)$, and $h(x)$ are all continuous in their domains; $a(x) > 1$ in U. ν is the outward unit normal on ∂U.

(a) Prove that a smooth solution to this problem is unique.

(b) Recall that $a(x) > 1$ in U. Suppose additionally that $f(x) \geq 0$ in U and $h(x) < 0$ on ∂U. Prove that $u \leq 0$ in $\overline{U}$.

3. [20 points] Solve the following problems.

(a) Let u and v be classical solutions of $u_t - u_{xx} = f(x,t)$ and $v_t - v_{xx} = g(x,t)$, respectively, on $\Omega = \{(x,t) \mid 0 < x < L,\ t > 0\}$ for some $L > 0$ fixed. Assume that

$$\begin{cases} f(x,t) \leq g(x,t) & \text{for all } (x,t) \in \Omega, \\ u(x,0) \leq v(x,0) & \text{for } 0 < x < L, \\ u(0,t) \leq v(0,t) \quad \text{and} \quad u(L,t) \leq v(L,t) & \text{for } t > 0. \end{cases}$$

Show that $u \leq v$ in Ω.

(b) Suppose that v is smooth and satisfies

$$v_t - v_{xx} \geq \sin(x) \qquad \text{in } Q = \{(x,t) \mid 0 < x < \pi,\ t > 0\}.$$

Moreover, assume that $v(0,t) \geq 0$ and $v(\pi,t) \geq 0$ for all $t \geq 0$ and $v(x,0) \geq \sin(x)$ for $0 \leq x \leq \pi$.

Show that $v(x,t) \geq \left(1 - \mathrm{e}^{-t}\right)\sin(x)$ in Q.

4. [20 points] Let $T > 0$ be fixed. Consider the following initial-boundary-value problem for the Kawahara equation

$$\begin{cases} u_t + uu_x + u_{xxx} + u_x + u_{xxxxx} = 0, & x \in (0,1), \quad t \in (0,T), \\ u(x,0) = \phi(x) & x \in (0,1), \\ u(0,t) = h_1(t),\ u(1,t) = h_2(t), & t \in (0,T), \\ u_x(0,t) = h_3(t),\ u_x(1,t) = h_4(t), & t \in (0,T), \\ u_{xx}(0,t) = h_5(t), & t \in (0,T). \end{cases} \tag{1}$$

where ϕ and h_j, $j = 1, 2, \ldots, 5$, are smooth functions. Show that the problem (1) admits only one smooth solution.

Hint: Grönwall's inequality states that if $y'(t) \leq g(t)y(t)$ for $t \geq 0$, then $y(t) \leq y(0)\,\mathrm{e}^{\int_0^t g(\tau)\,\mathrm{d}\tau}$.

Turn the page for problems 5 and 6.

5. [20 points] Use the method of characteristics to find the solution $u(x,t)$ of the Cauchy problem

$$\begin{cases} xu_x - u_y = u - 1, & -\infty < x < \infty, \quad y > 0, \\ u(x,0) = \sin(x), & -\infty < x < \infty. \end{cases}$$

6. [20 points] Find the entropy solution of the following problem

$$\begin{cases} u_t + uu_x = 0, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = g(x), & -\infty < x < \infty. \end{cases}$$

where

$$g(x) = \begin{cases} 1, & x < -1, \\ 0, & -1 < x < 1, \\ 1, & 1 < x. \end{cases}$$