Statistical Methods Prelim Exam

1:00 pm - 3:30 pm, Friday, May 5, 2023

UMVUE: uniformly minimum variance unbiased estimator; **i.i.d.**: identically and independently distributed; **UMP**: uniformly most powerful; **CRLB**: Cramér-Rao Lower bound; **LRT**: Likelihood Ratio Test; **r.s.**: random sample

1. Suppose that $X_1, X_2, \ldots, X_n$ be i.i.d $N(\mu_1, \sigma_1^2)$, and $Y_1, Y_2, \ldots, Y_n$ be i.i.d $N(\mu_2, \sigma_2^2)$ where $\mu_1, \mu_2 \in (-\infty, \infty)$, $\sigma_1^2 > 0$, and $\sigma_2^2 > 0$. Assume also and that X_i's and Y_i's, for $i = 1, .., n$, are independent.
   (a) Find the LRT statistic T based on X_i's and Y_i's $(i = 1, ..., n)$, for testing the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ versus the alternative hypothesis $H_1 : \sigma_1^2 \neq \sigma_2^2$. Give as simple an expression as possible for T.
   (b) Give an exact critical region of size α for the above test based on the suitable percentiles of a known distribution. Show work to fully justify your answer and identify the known distribution.

2. Let $X_1, \ldots, X_n$ be a r.s. from a normal population as $\mathcal{N}(\theta, \sigma^2)$, $\sigma^2 > 0$ is known, and suppose that the hypotheses to be tested are

$$H_0 : \theta \leq 0 \text{ versus } H_1 : \theta > 0$$

   (a) Derive a UMP test of size α for testing the above hypotheses.
   (b) Assume a prior distribution on θ is $\mathcal{N}(0, \tau^2)$, where τ^2 is known.
   Calculate the posterior probability that H_0 is true, i.e., $P(\theta \leq 0 | x_1, \ldots, x_n)$.
   (c) Show that $\lim_{\tau^2 \to \infty} P(\theta \leq 0 | x_1, \ldots, x_n) = p$-value for testing the hypotheses.
   (d) For the special case $\sigma^2 = \tau^2 = 1$ and for values $\overline{x} > 0$, compare the values of $P(\theta \leq 0 | x_1, \ldots, x_n)$ and the p-value of the test derived in Part (a) where $\overline{x}$ denotes the sample mean. Show that $P(\theta \leq 0 | x_1, \ldots, x_n)$ is always greater than the p-value.

3. Let $X_1, \ldots, X_n$ be a r.s. from the Poisson (θ) distribution truncated on the left at 0, with the sample space be the positive integers, $\mathbf{X} = \{1, 2, 3, \ldots\}$, and the probabaility mass function as $P(X = x) = e^{-\theta}\theta^x / [(1 - e^{-\theta})x!]$.
   (a) Derive the CRLB of the variance of unbiased estimators of θ.
   (b) Show that the CRLB is not attained by the UMVU estimator of θ.
   (c) Suppose that a random variable X (note: here $n = 1$) has the truncated Poisson(θ) distribution.
   Denote $\tau(\theta) = \exp(-\theta)$, the probability mass at zero for the untruncated Poisson distribution.
   Find the UMVUE of $\tau(\theta)$. Note $n = 1$ here. (Hint: The correct answer looks pretty silly, and is not a favorable choice for estimator in practice!)

4. Let $X_1, ..., X_n$ be a r.s. from a population with its density $f(x; \theta) = \theta(\theta + 1)x^{\theta - 1}(1 - x)$, $\theta > 0$ where $0 < x < 1$.
   (a) Show that $T_n = 2\overline{X}/(1 - \overline{X})$ is a method of moments estimate of θ where $\overline{X}$ denotes the sample mean.
   (b) Show that

$$\frac{\sqrt{n}(T_n - \mu_n(\theta))}{\sigma_n(\theta)} \xrightarrow{d} N(0, 1)$$

   where $\mu_n(\theta) = \theta$, $\sigma_n^2(\theta) = \theta(\theta + 2)^2 / 2(\theta + 3)$ and $\xrightarrow{d}$ denotes "convergence in distribution".
   (c) Show that T_n is not asymtotically efficient by calculating the information bound.