UNIVERSITY OF Cincinnati

MATHEMATICS QUALIFYING EXAM

MAY 3, 2018

Four Hour Time Limit

Notation: $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space. $\mathbb{N}$ is the set of natural numbers (positive integers).

Unless explicitly stated, proofs, or counterexamples, are required for all problems.

1. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le (x-y)^2$ for all $x, y \in \mathbb{R}$. Show that f is constant.

2. If
$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$
where $C_0, \ldots, C_n$ are real constants, prove that the equation
$$C_0 + C_1 x + C_2 x^2 + \ldots + C_{n-1}x^{n-1} + C_n x^n = 0$$
has at least one real root between 0 and 1.
Hint: Rolle's theorem.

3. Show that there is no one-one continuous function on $(0,1)$ for which $f((0,1)) = [0,2]$.

4. Let $f_n(x) = \frac{(n-1)x+x^2}{n+x}$ for all $x \ge 1$ and $n \in \mathbb{N}$. Let $f(x) = \lim_{n\to\infty} f_n(x)$ be the pointwise limit on $[1, \infty)$. Does $f_n(x) \to f(x)$ uniformly on $[1, \infty)$?

5. Let $\{\vec{v}_k : k \in \mathbb{N}\}$ be a set of non-zero vectors in some real (infinite dimensional) vector space. Suppose that for every $n \in \mathbb{N}$ vector $\vec{v}_{n+1}$ is not in the span of $\{\vec{v}_1, \ldots, \vec{v}_n\}$. Prove that the set $\{\vec{v}_k\}_{k\in\mathbb{N}}$ is linearly independent.

6. Consider a real vector space V of all twice-differentiable functions on $\mathbb{R}$ and its subspace U spanned by four functions $\{\sin x, \cos x, x\sin x, x\cos x\}$. Let $T : U \to V$ be a linear mapping given by $T(f) = f'' + f$.
(a) Determine the image of T.
(b) Determine the kernel of T.

7. Let A be an $n \times n$ matrix such that $A^3 = A^2 + A - I$.
(a) Show that A is invertible.
(b) Suppose, in addition, that A is diagonalizable. Show that A is its own inverse.

8. Let f be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ which send the point (x, y) into the point (u, v) given by
$$u = e^x \cos y,\ v = e^x \sin y.$$
Show that f is locally one-one at every point, but f is not one-one on $\mathbb{R}^2$.