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QUALIFYING EXAMINATION, AUGUST 2017

In this exam $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space. *This exam has eight questions.* Proofs, or counter examples, are required for all problems.

1. Let f, g and h be three functions on the interval $[-1, 1]$, and suppose that $f \leq h \leq g$ on this interval. Show that if f, g are differentiable at 0 with $f(0) = g(0)$ and $f'(0) = g'(0)$, then h is differentiable at zero with $h'(0) = f'(0)$.

2. Given a non-empty set $A \subset \mathbb{R}$, let $\rho_A : \mathbb{R} \to \mathbb{R}$ be given by

$$\rho_A(x) = \inf\{|x - z| \,:\, z \in A\}.$$

(a) Show that for all $x, y \in \mathbb{R}$ we have $|\rho_A(x) - \rho_A(y)| \leq |x - y|$.
(b) Using (a) above and the definition of absolute continuity, show that ρ_A is absolutely continuous on $\mathbb{R}$.
(c) Is ρ_A differentiable *everywhere*? Prove or give a counterexample (with a proof for counterexample).

3. Show that there are at least two distinct real numbers x for which

$$x^{10} + \frac{20}{1 + x^2 + \cos^2(x)} = 21.$$

Hint: You might want to consider the graph of $f(x) = x^{10} + \frac{20}{1+x^2+\cos^2(x)} - 21$.

4. Find all the positive integer values of n for which the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x, y) = \begin{cases} \frac{x^n y^n}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is differentiable at $(0, 0)$.

5. Let $f : [0, 1] \to \mathbb{R}$ be continuous, and let $\Gamma = \{(x, f(x)) \,:\, x \in [0, 1]\}$. State the definition of connectedness, and prove that Γ is a connected subset of $\mathbb{R}^2$. You may use without proof the fact that the interval $[0, 1]$ is a connected set.

6. In this question ℓ^1 is the collection of all sequences $\mathbf{x} = \{x_k\}_{k\in\mathbb{N}}$ of real numbers such that $\sum_{k=1}^{\infty} |x_k|$ is finite. Show that ℓ^1 is a vector space over the field $\mathbb{R}$, and that it is an infinite-dimensional vector space.

7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that for all vectors $\mathbf{x} \in \mathbb{R}^n$ we have $\|T(\mathbf{x})\| = \|\mathbf{x}\|$. Here, $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^n$.
(a) Show that if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with their inner product $\mathbf{x} \cdot \mathbf{y} = 0$, then $T(\mathbf{x}) \cdot T(\mathbf{y}) = 0$.
(b) Show that the columns of the matrix representing T in the standard basis of $\mathbb{R}^n$ form a mutually orthogonal collection of vectors.

Date: **The exam continues on the next page.** May 16, 2017.

8. For each positive integer n, denote by $\mathcal{P}_n$ the collection of all polynomials in x with coefficients in $\mathbb{R}$. You may take for granted that for each $\mathcal{P}_n$ is an $n+1$–dimensional vector space. Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be the linear map given by

$$Tp(x) = \frac{p(x) - p(0)}{x}.$$

(a) Show that T is surjective but not injective. You do not have to prove that T is a linear map.

(b) Show that the map $S : \mathcal{P}_2 \to \mathcal{P}_3$ given by $Sp(x) = x\,p(x)$ is a right inverse of T (that is, $T \circ S$ is the identity map on $\mathcal{P}_2$) but that S is *not* the inverse of T.