Preliminary Exam

Differential Equations

April 30, 2013

Name: **Student Id #:**

Instruction: Do all eight problems.

Score:

Problem 1.1 ————————- Problem 2.1————————

Problem 1.2 ————————- Problem 2.2————————

Problem 1.3————————- Problem 2.3————————

Problem 1.4————————- Problem 2.4————————

Part I total score : ——————- Part II total score ——————————

Total score ——————————

Part I: Ordinary Differential Equations

Problem 1.1

1. Define the operator norm $||T||$ of a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$.
2. Show that $|T(\mathbf{x})| \leq ||T|| \cdot |\mathbf{x}|$ for any linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ and any $\mathbf{x} \in \mathbb{R}^n$.
3. Given an $n \times n$ matrix A and $t \in \mathbb{R}$, define the matrix exponential e^{At} as a series. Show that for any fixed $t_0 > 0$ this series converges absolutely and uniformly for all $|t| \leq t_0$.

Problem 1.2: Solve the initial value problem

$$\begin{aligned} \dot{x}_1 &= 5x_1 - 3x_2 \\ \dot{x}_2 &= 3x_1 - x_2 \end{aligned}, \quad x_1(0) = 1, \quad x_2(0) = 2.$$

Problem 1.3: Consider the scalar initial value problem $\dot{x}(t) = 2tx^2, \quad x(0) = 1$.

(a) Solve the initial value problem exactly via separation of variables.

(b) Rewrite the system as an autonomous (i.e. no explicit time dependence) nonlinear initial value system by introducing $y(t) = t$ (be sure to determine $y(0)$).

(c) Use Picard iteration to compute the first four approximations $u_0(t)$, $u_1(t)$, $u_2(t)$, and $u_3(t)$ of the system you found in part (b). Compare your answer with the Maclaurin series for the function $x(t)$ you found in part (a).

Problem 1.4: Consider the system

$$\dot{x} = x^2 + a$$
$$\dot{y} = -y.$$

Determine the equilibria and their stability. Draw the bifurcation diagram. Draw the phase portraits for representative values of a.

Problem 2.1. Find the solution to

$$\begin{cases} u_t + 2x_1 u_{x_1} + x_2 u_{x_2} = u + x_1 & \text{in } \mathbf{R}^2 \times (0, \infty), \\ u(x_1, x_2, 0) = g(x_1, x_2) & \text{on } \mathbf{R}^2 \times \{t = 0\}. \end{cases}$$

Problem 2.2. Let $U \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. Prove that there does not exist any solution to the boundary-value problem

$$-\Delta u = 0 \ in \ U, \frac{\partial u}{\partial \nu} = 1 \ on \ \partial U.$$

Problem 2.3. Let Ω be a bounded domain in R^n with smooth boundary. Suppose u is a smooth solution of

$$\begin{cases} u_t - \Delta u + c(x,t)u = 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times [0,\infty) \\ u = g & \text{on } \Omega \times \{t=0\} \end{cases}$$

where $c(x,t)$ is a bounded function and $g \geq 0$.

(a) Assume additionally that $c(x,t)$ is nonnegative. Show that

$$u(x,t) \geq 0, \qquad \forall (x,t) \in \Omega \times [0,\infty).$$

(b) Is (a) still true without assuming $c(x,t)$ is nonnegative? If yes, give a proof; if no, give a counter-example.

Problem 2.4. Let $U \subset \mathbf{R}^n$ be open and bounded, with smooth boundary.

Show that a smooth solution to the PDE $\begin{cases} u_{tt} + \Delta u = 0 & \text{in } U \times [0,T] \\ u(x,t) = 0 & \text{on } \partial U \times [0,T] \\ u(x,0) = g(x) & \text{on } U \times \{t=0\} \\ u_t(x,0) = 0 & \text{on } U \times \{t=0\} \end{cases}$

satisfies the inequality $\displaystyle\int_U |Du|^2\,dx \geq \int_U |Dg|^2\,dx$ at every $t > 0$.