Preliminary Exam

Differential Equations

May 2, 2016

Name: **Student Id #:**

Instruction: Do all eight problems.

Score:

Problem 1.1 ——————-   Problem 2.1——————

Problem 1.2 ——————-   Problem 2.2——————

Problem 1.3——————   Problem 2.3—————-

Problem 1.4——————   Problem 2.4——————

Part I total score : —————-   Part II total score ——————

Total score ——————

Part I: Ordinary Differential Equations

Problem 1.1 Consider the system

$$\begin{aligned} x' &= x + \alpha\, y \\ y' &= x + y, \end{aligned}$$

where $\alpha \in \mathbb{R}$.

(a) For each value of α, identify if the equilibrium point $(x, y) = (0, 0)$ is a saddle, stable/unstable node, stable/unstable focus, center, or degenerate.

(b) For $\alpha = -1$, compute the solution to the system with initial condition $x(0) = 2$, $y(0) = 3$.

Problem 1.2 Assume that the functions $a(x) \geq 0$, and $u(x) \geq 0$ are continuous for $x \geq x_0$.

(a) Show that if

$$u(x) \leq \int_{x_0}^{x} a(t)u(t)dt \quad \text{for any } x \geq x_0$$

then $u(x) = 0$, for $x \geq x_0$.

(b) Show that if

$$u(x) \leq \int_{x_0}^{x} a(t)u^2(t)dt \quad \text{for any } x \geq x_0$$

then $u(x) = 0$, for $x \geq x_0$.

Problem 1.3 Use the appropriate Liapunov functions to determine the stability of the equilibrium point $(0, 0)$ of the following systems:

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + x_1 x_2 \\ \dot{x}_2 = x_1 - x_2 - x_1^2 - x_2^3 \end{cases}$$

Problem 1.4 Show that the system

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2)^2 \\ \dot{y} = x + y(1 - x^2 - y^2)^2 \end{cases}$$

has a semi-stable limit cycle Γ. Sketch the phase portrait for this system.

Problem 2.1. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open domain and $u(x)$ is a smooth function that satisfies

$$\begin{cases} \Delta u + x_1 u^2 u_{x_1} = 0 & \text{for all} \quad u \in \Omega, \\ u(x) = 0 & \text{for all} \quad x \in \partial\Omega. \end{cases}$$

Show that $u(x) = 0$ for all $x \in \Omega$.

Problem 2.2. Suppose $\Omega \subset \mathbb{R}^n$ is a connected, bounded open domain with smooth boundary Γ and $u(x)$ is a smooth function that satisfies

$$\Delta u = 0.$$

(a) Let ω any nonempty open subset of Ω. Show that $u(x) = 0$ for any $x \in \Omega$ if $u(x) = 0$ for any $x \in \omega$.

(b) Let Γ_0 be any nonempty open subset of Γ. Show that $u(x) = 0$ for any $x \in \Omega$ if

$$u|_{\Gamma_0} = 0, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_0} = 0.$$

Problem 2.3.

Find the entropy solution to the modified Burger's equation

$$\begin{cases} u_t + 3u^2 u_x &= 0 \quad \text{in } \mathbf{R} \times (0, \infty) \\ u &= g \quad \text{on } \mathbf{R} \times \{t = 0\} \end{cases}$$

with the initial data $g(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{if } x > 1 \end{cases}$.

Problem 2.4. Let u solves the in initial value problem for the wave equation in one dimension

$$\begin{cases} u_{tt} - u_{xx} = 0 & \textit{in } (-\infty, \infty) \times (0, \infty) \\ u = g, \quad u_t = h & \textit{on } (-\infty, \infty) \times \{t = 0\} \end{cases}$$

Suppose $g,\ h$ have compact support. Define

$$k(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x,t)dx, \qquad p(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x,t)dx$$

Prove that $k(t) = p(t)$ for all large enough time t.