Preliminary Exam

Differential Equations

August 16, 2017

Name: **Student Id #:**

Instruction: Do all eight problems.

Score:

Problem 1.1 ——————-	Problem 2.1——————
Problem 1.2 ——————-	Problem 2.2——————
Problem 1.3———————	Problem 2.3—————-
Problem 1.4———————	Problem 2.4——————

Part I total score : —————- Part II total score ———————

Total score ———————

Problem 1.1

Let A be an invertible 3×3 matrix, and consider the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$. Suppose there are three solutions $\mathbf{x}(t)$, $\mathbf{y}(t)$, $\mathbf{z}(t)$ with the properties

- $\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{0}$.
- $\lim_{t \to -\infty} \mathbf{y}(t) = \mathbf{0}$.
- $\mathbf{z}(4\pi) = \mathbf{z}(0)$.

Show that at least one of $\mathbf{x}(t)$, $\mathbf{y}(t)$, $\mathbf{z}(t)$ must be the constant solution at the origin.

Problem 1.2

The system of equations $\begin{cases} x' = x + 3\sin y \\ y' = x^2 + 4x - 3y \end{cases}$
has an equilibrium point at the origin $x = 0$, $y = 0$.

Determine whether the equilibrium is asymptotically stable, stable, or unstable.

Problem 1.3

Use the appropriate Lyapunov function to determine the stability of the equilibrium point of the system

$$\begin{cases} \dot{x}_1 = -2x_2 + x_2x_3 \\ \dot{x}_2 = x_1 - x_1x_3 \\ \dot{x}_3 = x_1x_2 \end{cases}$$

Problem 1.4

Consider the autonomous differential equation

$$v_{xx} + v - v^3 + v_0 = 0$$

in which v_0 is a constant.

a) Show that for $v_0^2 < \frac{4}{27}$, this equation has 3 stationary points and classify their type.

b) For $v_0 = 0$, draw the phase plane for this equation..

Problem 2.1.

Solve the following initial value problem.

$$u_x^2 + yu_y - u = 0 \text{ with the initial condition } u(x,1) = 1 + x^2/4.$$

Problem 2.2. Let $\Omega \subset R^n$ be a bounded regular domain. Consider a non-linear boundary value problem ($u \in C^{1,1}(\Omega)$)

$$\begin{cases} -\Delta u + \kappa_{(u>0)} = 0 \; in \; \Omega \\ u = \phi \; on \; \partial\Omega \end{cases}$$

where

$$\kappa_{(u>0)}(x) = \begin{cases} 1 \; if \; u(x) > 0, \\ 0 \; if \; u(x) \leq 0. \end{cases}$$

Prove that $u(x) \geq 0$ in Ω if $\phi > 0$ on $\partial\Omega$.

Problem 2.3.

(i) Show that if a function $u \in C(\Omega)$ satisfies the mean value property for each ball $B(x,r) \subset \Omega$, then $u \in C^\infty(\Omega)$.

(ii) Let $\{u_n\}_{n=1}^\infty$ be a sequence of harmonic functions in $C(\Omega)$. If u_n is uniformly convergent to u in Ω as $n \to \infty$, then u is also harmonic function in Ω.

Problem 2.4.

Fix a number $L > 0$ and consider the initial-boundary value problem of the linear six-order Boussinesq equation

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} - u_{xxxxxx} = 0 \text{ in } (0,L)\times(0,T), \\ u(x,0) = g(x) \text{ and } u_t(x,0) = h(x), \\ u(0,t) = 0,\ u(L,t) = 0,\ u_{xx}(0,t) = 0,\ u_{xx}(L,t) = 0,\ u_{xxxx}(0,t) = 0,\ u_{xxxx}(L,0) = 0 \end{cases} \tag{*}$$

i) Define $E(t) = \displaystyle\int_0^L \left(u_t^2(x,t) + u_x^2(x,t) + u_{xx}^2(x,t) + u_{xxx}^2(x,t)\right) dx$. Show that

$$E(t) = \int_0^L \left(h^2(x) + (g'(x))^2 + (g''(x))^2 + (g'''(x))^2\right) dx$$

for any $0 \le t \le T$.

ii) Show that (*) admits at most one smooth solution..