UC Calculus Contest

April 7, 2015

Name:____________________ **M#:**______________ **Instructor:**_________________

Instructions: *This exam has seven problems on seven pages. Show all your work, expressing yourself in clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.*

1

Let S_1 be the 1×1 square. Define by induction the squares S_{i+1}equal to the square obtained by connecting the midpoints of the sides of the square S_i. Find $\sum_{i=1}^{\infty}$perimeter (S_i) and $\sum_{i=1}^{\infty}$area (S_i).

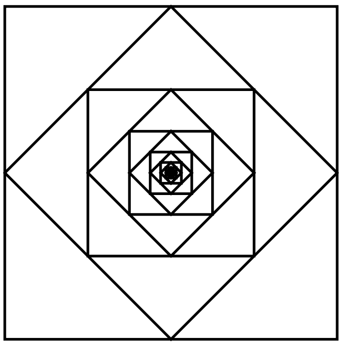

Figure 1.

2

Suppose that a, b are nonzero real numbers and f is differentiable at x. Express the limit

$$\lim_{h \to 0} \frac{f(a\,h + x) - f(b\,h + x)}{h}$$

In terms of a, b and $f'(x)$.

3

Consider the series

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n}$$

(a) Show that the series converges.

(b) Find the sum of the series.

4

Let x be a real number. Show that the limit exists, and then find it:

$$\lim_{n\to\infty} \sin(\ \ldots \sin(\sin(\sin(\sin(x))))$$

(above, the sin is calculated n times).

5

Assuming that $a > -1$ and $b > -1$, use Riemann sums to calculate

$$\lim_{n\to\infty} n^{b-a} \frac{1^a + 2^a + 3^a + ... + n^a}{1^b + 2^b + 3^b + ... + n^b}.$$

6

It is known (to be proved in the Multivariable Calculus) that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\,.$$

Show that

$$\int_{1}^{\infty} \left(\frac{1}{x}\right)^{\ln(x)} dx = \frac{\sqrt{\pi}}{2}\, e^{1/4} \left(\operatorname{erf}\left(\frac{1}{2}\right) + 1\right)$$

where

$$\operatorname{erf}(z) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2}\, dt.$$

7

Show that series

(a)

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\ln(n)}$$

and

(b)

$$\sum_{n=1}^{\infty}\left(2^{1/n}-1\right)^{\ln(n)}$$

are both convergent.