ANALYSIS PRELIMINARY EXAMINATION, SPRING 2018

Real Analysis

In this part of the exam, m or dx (resp., m^2) denote Lebesgue measure on $\mathbb{R}$ (resp., on $\mathbb{R}^2$).

(1) Carefully justifying your answer, evaluate:

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin x}{1+n^2x^2}\, dx.$$

(2) Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions. Show that the set

$$\{x \in \mathbb{R}\colon (f_n(x))_{n=1}^\infty \text{ converges to a real number}\}$$

is measurable. Hint: a sequence in $\mathbb{R}$ converges if and only if it is Cauchy.

(3) Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous strictly increasing function. Prove that for every $\epsilon > 0$ there is $\delta > 0$ such that if $E \subset [0,1]$ and $m^*(E) < \delta$, then $m^*(f(E)) < \epsilon$, where m^* denotes the Lebesgue outer measure.

(4) Let $f \in L^1(0,\infty)$. For $x > 0$, define $g(t,x) = tf(t)e^{-tx}$. Prove that $g \in L^1((0,\infty)\times(0,\infty))$ and

$$\int_{(0,\infty)\times(0,\infty)} g(t,x)\, dm^2(t,x) = \int_0^\infty f(t)\, dm(t)$$

justifying all your steps.

Complex Analysis

In this part of the exam, $\mathbb{C}$ denotes the collection of all complex numbers.

(1) Compute the following integral using the method of residues or the argument principle:

$$\int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+9)}\, dx.$$

(2) Let f be given by $f(z) = \frac{z}{1+z}$ and for each positive integer n let the function g_n be the n-fold composition of f with itself, so $g_2 = f\circ f$, $g_3 = f \circ f \circ f$, etc.
  (a) Find an explicit formula for g_n for each positive integer n.
  (b) Prove that the sequence g_n converges to zero uniformly on the disk $\{z : |z-1| < 1\}$.

(3) Let $a, b \in \mathbb{C}$ with $a \neq b$, and let $F(z) = \frac{z-a}{z-b}$.
  (a) Find the image of the line passing through a and b and ∞.
  (b) Find the image of the perpendicular bisector of the line $[a,b]$ (including ∞ as a point in that line).
  (c) Find the image of the Euclidean circle centered at $(a+b)/2$ with radius $|a-b|/2$ (that is the circle centered at the midpoint between a and b, and passing through both a and b).

(4) Let f and g be two non-constant holomorphic (that is, complex analytic) functions in a region $\Omega \subset \mathbb{C}$ such that $|f(z)| \leq |g(z)|$ for all $z \in \Omega$. Let $K = g^{-1}(\{0\})$. Prove that the function f/g is analytic on $\Omega \setminus K$ and that it has an analytic extension to all of Ω. Use this to prove that if F is an holomorphic function on $\mathbb{C}$ with $|F(z)| \leq |\sin(\pi z)|$ for all $z \in \mathbb{C}$ then there is some complex number c with $|c| \leq 1$ such that $F(z) = c\sin(\pi z)$ for all $z \in \mathbb{C}$.