PhD Preliminary Exam in Algebra and Topology
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Full credit can be obtained by complete answers to 5 questions, of which at least two must come from each section. The examination lasts four hours.

Algebra

(1) Denote by $\mathbb{R}(x)$ the field of fractions of the ring $\mathbb{R}[x]$ of polynomials with real coefficients. Let S be the subring of $\mathbb{R}(x)$ consisting of those fractions whose denominators are relatively prime to x^2+1. That is,

$$S = \{p(x)/q(x) \in \mathbb{R}(x) \mid \gcd(q(x), x^2+1) = 1\}.$$

(a) What are the units of S?
(b) Identify the ideals of S.
(c) Is S a unique factorization domain? Explain.
(d) If $\mathbb{R}$ is replaced by $\mathbb{C}$ and the set of rational functions corresponding to S constructed, would it have a unique maximal ideal? Explain.

(2) Let $K \supset F$ be fields such that K is a finite Galois extension of F and suppose that $Gal(K, F) = S_4$. How many proper intermediate fields are there between F and K? Which of these intermediate fields are Galois extensions of F and what are their Galois groups?

(3) Let $f(x) = x^5 - 2 \in \mathbb{Q}[x]$.
(a) Find a splitting field for f over $\mathbb{Q}$..
(b) Find the Galois group for f.
(c) Find all proper, nontrivial normal subgroups of this Galois group and the fields to which they correspond according to the fundamental theorem of Galois theory.

(4) Let p be a prime number, q a power of p. Let $F \subset K$ be fields such that $|F| = p$ and $|K| = q$. Let f be an irreducible polynomial in $F[x]$. Prove that any two irreducible factors of f over the field K have the same degree.

Topology

(1) Let X be the countable product of the real line equipped with the box topology.
(a) Is X Hausdorff?
(b) Is X connected?
(c) Does X have a countable dense set?
Justify your answer.

(2) Let D^n be the n-dimensional ball and S^{n-1} the $(n-1)$-dimensional sphere, which is the boundary of D^n. Prove that the following are equivalent.
(a) There is no retraction from D^n to S^{n-1}.
(b) Every continuous map from D^n to D^n has a fixed point.

(3) Let G be a finitely generated abelian group. Find a finite dimensional path-connected topological space X with $\pi_1(X) = G$.

(4) Let $\mathbf{RP}^2$ be the real projective plane; let X be the one point union $S^1 \vee \mathbf{RP}^2$.
   (a) Compute $\pi_1(X)$.
   (b) Find the universal covering space of X.