PhD Preliminary Exam in Algebra and Topology
May 2015
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Full credit can be obtained by complete answers to 5 questions, of which at least two must come from each section. The examination lasts four hours.

Algebra

Let $\mathcal{Q}$ be the field of rational numbers and $\mathcal{Z}$ be the ring of integers.

(1) Let $f(x) = x^8 + x^4 + 1$ be a polynomial in $\mathcal{Q}[x]$. Suppose E is a splitting field for $f(x)$ over $\mathcal{Q}$ and set $G = Gal(E/\mathcal{Q})$.
  (a) Find $|E : \mathcal{Q}|$ and determine the Galois group G up to isomorphism.
  (b) If $\Omega \subset E$ is the set of roots of $f(x)$, find the number of orbits for the action of G on Ω.

(2) Prove that if an integer polynomial $f(x)$ of positive degree is irreducible in $\mathcal{Z}[x]$ then it is also irreducible in $\mathcal{Q}[x]$. Use this to prove that $\mathcal{Z}[x]$ is a unique factorization domain.

(3) Let E be the field $E = \mathcal{Q}[\sqrt[3]{2}, \sqrt{2}]$. Find $|E : \mathcal{Q}|$. Find an element α such that $E = \mathcal{Q}[\alpha]$. Find the irreducible polynomial $f(x)$ in $\mathcal{Z}[x]$, for which α is a root. Is $E \supset \mathcal{Q}$ a Galois extension? Prove your statement.

(4) Let F_q be a finite field and E a degree n extension of F_q. Prove that this is a Galois extension and give an explicit description of the Galois group.

Topology

(1) Suppose X is Hausdorff, $A \subset X$ is compact, and $f : X \to X$ is continuous. Prove that the set $\{x \in A | f(x) \in A\}$ is also compact.

(2) If $A \subset X$ is a subset of a topological space we will let A' denote the limit points of A.
  (a) Define *limit point.*
  (b) If $A \subset X$ an $B \subset Y$ prove that in $X \times Y$ it is always the case that $A' \times B' \subset (A \times B)'$.
  (c) Give an example that shows equality may not hold in the above containment.

(3) Let D^n be a standard n-dimensional ball and let S^{n-1} be the boundary of D^n. Prove that the following are equivalent.
  (a) There is no retraction from $D^n \to S^{n-1}$.
  (b) Every continuous map from $D^n \to D^n$ has a fixed point.

(4) Let X be path connected, $p : E \to Y$ be a covering map, and $f : X \to Y$ be continuous.

Prove the following: If E is simply connected, and the image of $f_* : \pi_1(X) \to \pi_1(Y)$ is nontrivial, then there does not exist a lifting $F : X \to E$ such that $p \circ F = f$.