Algebra/Topology Preliminary Exam
April 2018

I. Algebra

(1). Let R be an integral domain.

(a) Define what it means for an element $r \in R$ to be *irreducible*.
(b) Define what it means for an element $r \in R$ to be *prime*.
(c) Show that in an integral domain a prime element is irreducible.
(d) Show that in a principal ideal domain that an irreducible element is prime.

(2). Suppose $F = \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i^2 \in \mathbb{Q}$ for $i = 1, 2, \cdots, n$. Prove that $\sqrt[3]{3} \notin F$.

(3). Let F be a field.

(a) Let $\alpha \in F$ be algebraic. Prove there is a unique monic irreducible polynomial $m_\alpha(x) \in F[x]$ that has α as a root.
(b) Prove that $\alpha \in F$ is algebraic if and only if $F(\alpha)/F$ is a finite extension.

(4). Let $F = \mathbb{Q}(\sqrt{3}, \sqrt{5})/\mathbb{Q}$.

(a) Prove that F is a Galois extension.
(b) Compute the Galois group.
(c) Explicitly give the correspondence between the subfields of F and the subgroups of the Galois group.

II. Topology

(1). Let X and Y be topological spaces and suppose $f : X \to Y$. Show the following three conditions are equivalent:

(i) f is continuous.
(ii) For every subset $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
(iii) For every closed set $B \subset Y$, the set $f^{-1}(B)$ is closed in X.

(2). Let A and B be subspaces of X and Y, respectively. Let N be an open set in $X \times Y$ containing $A \times B$. Suppose A and B are compact. Show there exist open sets U and V in X and Y, respectively, such that $A \times B \subset U \times V \subset N$.

(3). Let X be a topological space.

(a) Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.
(b) Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

(4). Let $q : X \to Y$ and $r : Y \to Z$ be covering maps. Set $p := r \circ q$. Show that if $r^{-1}(z)$ is finite for each $z \in Z$, then p is a covering map.