Topology Preliminary Exam, May 2025

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To get full credit for a problem, make sure to provide a complete proof for all your claims in your answer. The set of all real numbers is denoted by $\mathbb{R}$ in this exam.

(1) Let $(X, \mathcal{T})$ be a topological space and E, F be non-empty connected subsets of X.
  (a) Prove that if $E \cap F$ is non-empty, then $E \cup F$ is connected.
  (b) Prove that if E and F are both closed and $E \cup F$ is connected, then $E \cap F$ is non-empty.
  (c) Give a counterexample topological space X and connected subsets E and F of X such that $E \cup F$ is connected but $E \cap F$ is empty.

(2) Let $\mathcal{T}$ be the collection of all sets $U \subset \mathbb{R}$ (with $\mathbb{R}$ the set of all real numbers) such that either U is empty or $0 \in U$.
  (a) Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$.
  (b) Let $A = \{1/n : n \in \mathbb{N}\}$, where $\mathbb{N}$ is the collection of all positive integers, and let $E = \{0\}$. What are the closure of A and the closure of E in this topology?
  (c) Is $\mathbb{R}$ path-connected in this topology? Prove your assertion.

(3) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces with Y compact. Let $X \times Y$ be equipped with the corresponding product topology. Let $\pi : X \times Y \to X$ be the projection map given by $\pi(x, y) = x$. Prove that if E is a closed subset of $X \times Y$, then $\pi(E)$ is a closed subset of X.

(4) Let $X = \mathbb{R}^2$, and S a fixed collection of polynomials in two variables. We also set $Z := \{(x, y) \in \mathbb{R}^2 : \forall f \in S \text{ we have } f(x, y) = 0\}$. Let $\mathcal{U}$ consist of all sets $U \subset \mathbb{R}^2$ for which $\mathbb{R}^2 \setminus U \subset Z$.
  (a) Prove that $\mathcal{U}$ is a basis for a topology on $\mathbb{R}^2$.
  (b) Is this topology metrizable?

(5) Let $\mathbb{R}^3$ be equipped with the Euclidean topology, and let $X = \mathbb{R}^3 \setminus \mathbb{S}^1$ be equipped with the subspace topology. What is the fundamental group of X? Here $\mathbb{S}^1$ is the unit circle $\mathbb{S}^1 := \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.