Preliminary Exam

Partial Differential Equations — May 2025

Please start each problem on a new page, and arrange your work in order when submitting.
Please choose 5 problems out of the 6 problems on the exam. Only 5 problems will be graded.

1. Consider the following nonlinear (Dirichlet) boundary-value problem on a (open and connected) domain $D \subset \mathbb{R}^n$ with smooth boundary:

$$\begin{cases} \Delta u - u^2 = f(x) & \text{in } D, \\ u = g(x) & \text{on } \partial D, \end{cases}$$

with given smooth functions $f(x)$ and $g(x)$. Show that it is impossible to have two distinct positive solutions of this problem.

2. Let R be the square $R = \{(x,y)\colon -1 < x < 1, -1 < y < 1\} \subset \mathbb{R}^2$. Assume that $u(x,y)$ satisfies

$$\begin{cases} \Delta u = -1 & \text{in } R, \\ u = 0 & \text{on } \partial R. \end{cases}$$

Show that $\frac{1}{4} \leq u(0,0) \leq \frac{1}{2}$.
Hint: Consider $v(x,y) = u(x,y) + \frac{1}{4}\left(x^2 + y^2\right)$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (open and connected) with smooth boundary. Assume that $u(x,t)$ is a smooth function on $\Omega \times [0,\infty)$ solving the initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u + V(x)u = h(x), & x \in \Omega, \quad t > 0, \\ u(x,0) = f(x), & x \in \Omega, \\ u_t(x,0) = g(x), & x \in \Omega, \\ u + \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \quad t \geq 0, \end{cases} \tag{1}$$

where n denotes the outward unit normal vector on $\partial\Omega$ and $f(x), g(x), V(x)$, and $h(x)$ are smooth functions on Ω.

   (a) **For this part only, suppose that** $h \equiv 0$**. In this case,** show that

$$E(t) = \frac{1}{2}\int_\Omega \left((u_t(x,t))^2 + |\nabla u(x,t)|^2 + V(x)(u(x,t))^2\right) \mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} (u(x,t))^2 \,\mathrm{d}S(x)$$

   is a conserved quantity, i.e., $E(t) \equiv$ constant for all $t \geq 0$. What is the value of this constant (in terms of the data given in the problem (1))?

   (b) Use part (a) to show that for any smooth f, g, h, V with $V(x) \geq 0$ on Ω, the initial-boundary value problem has at most one smooth solution.

The exam continues on the next page.

4. Let Ω be a bounded domain (open and connected) in $\mathbb{R}^n$. Assume that g is a function continuous in $\overline{\Omega} \times [0, T]$, u_0 is a function continuous in $\overline{\Omega}$ with $u_0 \geq 0$, and $F : \mathbb{R} \to \mathbb{R}$ is continuous with $F(s) \leq C < +\infty$ for some constant $C > 0$ for all $s \in \mathbb{R}$.

   Suppose that $u = u(x,t)$ with $u \in C^{2,1}(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])$ is a solution of

$$\begin{cases} u_t - \Delta u + gu = uF(u) & \text{in } \Omega \times (0,T], \\ u(\cdot, 0) = u_0 & \text{on } \Omega \times \{0\}, \\ u = 0 & \text{on } \partial\Omega \times (0,T]. \end{cases}$$

   Prove that

$$0 \leq u(x,t) \leq \sup_{\Omega} u_0, \quad \text{for all } (x,t) \in \Omega \times (0,T].$$

   Hint: Work with $w = u\mathrm{e}^{-Mt}$ for a suitable choice of a constant M.

5. Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon x_2 > 0\}$ and $\Gamma = \partial\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon x_2 = 0\}$. Solve the boundary value problem

$$\begin{cases} u_{x_1} + 4u_{x_2} = u^2 & \text{in } \Omega, \\ u(x) = \sin(x_1) & \text{on } \Gamma. \end{cases}$$

   Find the smallest value of $x_2^* > 0$ such that the solution forms a singularity at $x = (x_1, x_2^*)$ for some $x_1 \in \mathbb{R}$. For such smallest value of $x_2^* > 0$, list values of x_1 for which the singularity is formed.

6. Find the entropy solution $u = u(x,t)$ for the initial-value problem

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

   with the initial data

$$g(x) = \begin{cases} 2, & x < -1, \\ 0, & -1 < x < 1, \\ 1, & x > 1. \end{cases}$$