Linear algebra qualification exam

May 5, 2025

1. (a) Let A be an $m \times n$ matrix with linearly independent columns. Show that the matrix $A^T A$ is square, symmetric, and invertible. (A^T denotes transpose.)

   (b) Show that the eigenvalues of $A^T A$ are positive.

2. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be such that Px rotates x by an angle θ counterclockwise and reflects the result with respect to the x_1-axis.

   (a) Show that P is a linear transformation.

   (b) Show that $PPx = x$ for any $x \in \mathbb{R}^2$.

   (c) Find the matrix for P (with respect to the standard basis).

3. Suppose U_1, U_2, and W are finite dimensional subspaces of a vector space V with the properties

   - $U_1 \cap U_2 = \{\mathbf{0}\}$.
   - $U_2 \subseteq U_1 + W$.

   Show that $\dim U_2 \leq \dim W$.

4. Consider the four functions $f_1(x) = \sin x$, $f_2(x) = \sin 2x$, $f_3(x) = \cos x$, $f_4(x) = \cos 2x$. Let $V = \operatorname{span}(f_1, f_2, f_3, f_4)$ and let $T : V \to V$ be the differentiation map $Tf = f'$.

   (a) What is the null space of T?

   (b) Write out the matrix for T with respect to the basis (f_1, f_2, f_3, f_4).

5. Continue with the notation of Problem 4 above. Define an inner product on V by

$$\langle f, g \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x) g(x) dx.$$

(a) Prove that $\mathcal{B} = \{f_1, f_2, f_3, f_4\}$ is an orthonormal basis of V. You can take for granted that $\int_0^{2\pi} \sin^2(x)dx = \pi$. Use periodicity to minimize the amount of computation.

(b) Compute the coordinates of $(\cos x + \sin x)^2 - 2\sin^2 x$ with respect to $\mathcal{B}$.