**This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted. Proofs or counterexamples are required for all problems. Time allowed: 2 hours 30 minutes.**

**Notation: $\mathbb{R}$ is the real line; $\mathbb{R}^2$ is the plane $\mathbb{R} \times \mathbb{R}$; $\mathbb{N}$ is the set of positive integers.**

1. (a) Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of non-negative numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that the series $\sum_{n=1}^{\infty} a_n^2$ also converges.

   (b) Is the converse of the statement in part (a) true? Prove or give a counterexample.

2. Let $A \subset \mathbb{R}^2$ be a compact set. Prove that the set $\{x \in \mathbb{R} \colon \exists\, y \in \mathbb{R} \text{ such that } (x, y) \in A\}$ is also compact.

3. **Definition:** For a number $L > 0$, a function $f \colon \mathbb{R} \to \mathbb{R}$ is called Lipschitz with constant $L$ if

$$|f(x) - f(y)| \leq L|x - y|, \quad \text{for all } x, y \in \mathbb{R}.$$

   Assume that a function $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$. Prove that $f$ is Lipschitz with constant $L$ if and only if $|f'(x)| \leq L$ for all $x \in \mathbb{R}$.

4. Let the function $f \colon [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x = \frac{1}{n},\ n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

   Prove or disprove: $f$ is Riemann integrable on the interval $[0, 1]$.

5. (a) Give the definition of what it means for a function $g \colon \mathbb{R} \to \mathbb{R}$ to be uniformly continuous on a set $S \subset \mathbb{R}$.

   (b) Let $g \colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous on a set $S \subset \mathbb{R}$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence of points in $S$. Prove that the sequence $\{g(x_n)\}_{n \in \mathbb{N}}$ is also Cauchy.

6. (a) Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sin(2^n x)}{2^n}$$

   converges for all $x \in \mathbb{R}$.

   (b) Let $h \colon \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin(2^n x)}{2^n}$. Prove that $h$ is continuous on $\mathbb{R}$.