Ordinary Differential Equation Preliminary Exam

January 9, 2025

Answer any five out of the six questions.

Problem 1

(a) Find the general solution of the system (here $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$)

$$x' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x.$$

(b) Draw the integral curves near the origin, and indicate the direction in which they are traveled.

Problem 2

(i) Show that the rest point $(0,0)$ is asymptotically stable for the system

$$\begin{aligned} x' &= -5y - x(x^2+y^2) \\ y' &= x - y(x^2+y^2)\,, \end{aligned}$$

and that its domain of attraction is the entire xy-plane.

(ii) Draw the integral curves near the origin, and indicate the direction in which they are traveled.

Problem 3 Solve the initial value problem

$$\begin{cases} \frac{dx_1}{dt} = -2x_2 - x_3 - x_4 \\ \frac{dx_2}{dt} = x_1 + 2x_2 + x_3 + x_4 \\ \frac{dx_3}{dt} = x_2 + x_3 \\ \frac{dx_4}{dt} = x_4 \\ x_1(0) = x_2(0) = x_3(0) = x_4(0) = 1 \end{cases}$$

Problem 4 Write the following system in polar coordinates and determine if the origin is a center, a stable focus or an unstable focus.

$$\frac{dx}{dt} = -y + xy^2, \quad \frac{dy}{dt} = x + y^3.$$

Problem 5 Consider the equation $y''' - 2y'' - y' + 2y = 0$.

(i) Write this linear differential equation in the form of $x' = Ax$;

(ii) For the system $x' = Ax$, determine the stable subspace E_s, the unstable subspace E_u,and the center subspace E_c, if applicable;

(iii) Sketch the phase portrait of this system.

Problem 6 Use Lyapunov's method to study the stability of the zero solution for the planar system of equations

$$x' = y^2 - x^3, \quad y' = -y - 2xy.$$