Topology Preliminary Exam, August 2025

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To get full credit for a problem, make sure to provide a complete proof for all your claims in your answer. The set of all real numbers is denoted by $\mathbb{R}$ in this exam. You may use without proof the fact that intervals in $\mathbb{R}$ are connected with respect to the Euclidean topology.

(1) Let $X = \{(x, y) : x, y \in \mathbb{R}\}$, and let $\mathcal{C}$ be the collection of all sets $K \subset X$ for which there is a (possibly empty) collection S_K of polynomials in x, y such that

$$K = \{(x, y) \in X : f(x, y) = 0 \text{ for each } f \in S_K\}.$$

Prove that the collection $\mathcal{T} := \{X \setminus K : K \in \mathcal{C}\}$ is a topology on X. (This is called the Zariski topology on $\mathbb{R}^2$.)

(2) Let $(X, \mathcal{T}_X)$ be a compact topological space and $(Y, \mathcal{T}_Y)$ be a Hausdorff space. Suppose that $f : X \to Y$ is a continuous bijective map. Prove that $f(U) \in \mathcal{T}_Y$ for each $U \in \mathcal{T}_X$.

(3) Let $\mathcal{T}$ be the collection of all subsets U of $\mathbb{R}$ for which either $U = \mathbb{R}$ or else $\mathbb{R} \setminus U$ is a finite set. Let $A = \{1/n : n \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of all positive integers. Identify the closure of A and the boundary ∂A of A with respect to the topology $\mathcal{T}$ on $\mathbb{R}$. You may use the fact that $\mathcal{T}$ is a topology without proof.

(4) Let $\mathcal{T}_{\text{Euc}}$ be the Euclidean topology on $\mathbb{R}$, and let $X = [-1, 1]$. Let $\mathcal{T}$ be the collection of all sets $U \subset X$ for which either $U \in \mathcal{T}_{\text{Euc}}$ or else $\{-1, 1\} \subset U$ and $U = [-1, 1] \cap W$ for some $W \in \mathcal{T}_{Euc}$.
   (a) Show that $\mathcal{T}$ is a topology on X.
   (b) Show that $X \setminus \{0\}$ is connected with respect to this topology.

(5) Let $\mathbb{C}$ denote the collection of all complex numbers, equipped with the Euclidean topology, and let $\mathbb{C}^*$ be the collection of all non-zero complex numbers, equipped with the Euclidean subspace topology. Let $f : \mathbb{C} \to \mathbb{C}^*$ be given by $f(z) = e^z$.
   (a) Prove that f is a covering map.
   (b) Compute the fundamental group of $\mathbb{C}^*$.