# Real Analysis Preliminary Exam, August 21, 2025

Time allowed: 2 hours 30 minutes.

This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted.

**Notation:** $\mathbb{R}$ is the set of real numbers; $m$ is the Lebesgue measure on $\mathbb{R}$; $m^*$ is the Lebesgue outer measure on $\mathbb{R}$.

1. Let $A_1$ and $A_2$ be two disjoint Lebesgue measurable subsets of $\mathbb{R}$. Let $B_1 \subset A_1$ and $B_2 \subset A_2$. Prove that $m^*(B_1 \cup B_2) = m^*(B_1) + m^*(B_2)$.

2. Let $\mu$ be a non-negative measure on the measurable space $(\mathbb{R}, \mathcal{L})$, where $\mathcal{L}$ is the $\sigma$-algebra of Lebesgue measurable subsets of $\mathbb{R}$. Assume that there exists $M \geq 0$ such that
$$\int_{\mathbb{R}} e^{nx^3}\, d\mu \leq M, \quad n = 1, 2, 3, ...$$
   (a) Prove that $\mu((0, \infty)) = 0$.
   (b) Is it necessarily true that $\mu([0, \infty)) = 0$? Prove or give a counterexample.

3. Let $A_1, \ldots, A_n \subset [0, 1]$ be Lebesgue measurable sets such that $\sum_{k=1}^{n} m(A_k) > 4$. Prove that there is a point $x \in [0, 1]$ that lies in at least 5 of these sets.

4. Let $f$ be a Lebesgue integrable function on $(0, \infty)$. Prove that
$$\lim_{n\to\infty} \frac{1}{n^3} \int_0^n x^3 f(x)\, dm = 0.$$

5. (a) Give the definition of a function of bounded variation on $[0, 1]$.
   (b) Let $f$ be a function of bounded variation on $[0, 1]$. Prove that $f$ is Lebesgue measurable on $[0, 1]$.

6. Let $f$ be a Lebesgue integrable function on $(1, \infty)$. Let $g(x) = \int_x^\infty \frac{f(t)}{t}\, dt$ for $x > 1$. Prove that $g$ is Lebesgue integrable on $(1, \infty)$.