Linear algebra qualification exam

August 20, 2025

1. Let $U, W \subseteq V$ be subspaces of a finite-dimensional vector space. Suppose $\dim U = \dim W$.

   (a) Show that there exists an invertible linear transformation $S \in \mathcal{L}(V)$ so that $S\big|_U$ is an isomorphism between U and W.

   (b) Suppose that $T \in \mathcal{L}(V)$, and that $\dim(\text{null } T) \geq \dim(\text{range } T)$. Show that there exists an invertible linear transformation $S \in \mathcal{L}(V)$ so that $TSTv = \vec{0}$ for all $v \in V$.

2. Let $T \in \mathcal{L}(\mathbb{R}^n)$, and equip $\mathbb{R}^n$ with the Euclidean norm $|v| = \sqrt{v \cdot v}$.

   Suppose T has the property than $\lim_{k \to \infty} |T^k v| = 0$ for all $v \in \mathbb{R}^n$.

   (a) Prove that every eigenvalue of T satisfies $|\lambda| < 1$, or give an example where this is not true.

   (b) Prove that $|Tv| < |v|$ for all $v \in \mathbb{R}^n$, or give an example where this is not true.

3. Let V be a complex vector space of finite dimension n, and $T \colon V \to V$ be a linear operator.

   (a) Prove that any such T has a (complex) eigenvector.

   (b) Using (a), prove that T has a matrix representation of the form

   $$\begin{pmatrix} \lambda & x^T \\ 0 & M \end{pmatrix},$$

   for some $\lambda \in \mathbb{C}$, $x \in \mathbb{C}^{n-1}$, and M is a $(n-1) \times (n-1)$ matrix with complex entries.

4. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if S and T are invertible.

5. Let $(V, \langle \cdot, \cdot \rangle)$ be a complex inner product space.

   (a) Prove that if $T = S^*S$ for some invertible $S \in \mathcal{L}(V)$, then $I_T(v, w) := \langle v, Tw \rangle$ is an inner product too.

   (b) Is the converse true? (Hint: first try to decide if T is self-adjoint.)