This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted. Proofs or counterexamples are required for all problems. Time allowed: 2 hours 30 minutes.

Notation: $\mathbb{R}$ is the real line; $\mathbb{N}$ is the set of positive integers.

1. (a) Prove the Mean Value Theorem for integrals: If f is a continuous real-valued function on an interval $[a,b]$, then there exists $c \in (a,b)$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

   (b) Is the statement in part (a) true if "continuous" is replaced with "Riemann integrable"? Prove or give a counterexample.

2. Prove the Cauchy Condensation Test: if $\{a_n\}$ is a decreasing sequence of positive numbers, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converges.}$$

3. Let B be a bounded subset of $\mathbb{R}$ such that every Cauchy sequence of points in B converges to a point in B. Prove that B is compact.

4. Suppose that $f : [0,1] \to \mathbb{R}$ is increasing and let $a \in (0,1)$.
   (a) Prove that $\lim_{x\to a^-} f(x) = \sup\{f(x) \colon 0 \le x < a\}$.
   (b) Prove that $\lim_{x\to a^+} f(x) = \inf\{f(x) \colon a < x \le 1\}$.
   (c) Prove that $\sup\{f(x) \colon 0 \le x < a\} = \inf\{f(x) \colon a < x \le 1\}$ if and only if f is continuous at a.

5. **Definition:**
   A function $f \colon [c,d] \to \mathbb{R}$ is called Lipschitz if there exists a number $L > 0$ such that

$$|f(x) - f(y)| \le L|x-y|, \quad \text{for all } x, y \in [c,d].$$

   Let $g \colon [a,b] \to [c,d]$ be a Riemann integrable function on $[a,b]$ and $f \colon [c,d] \to \mathbb{R}$ be a Lipschitz function. Prove that the composition $f \circ g$ is Riemann integrable on $[a,b]$.

6. Let $g_n(x) = n(1-x)x^n$ for $n \in \mathbb{N}$ and $x \in [0,1]$.
   (a) For all $x \in [0,1]$, compute the limit function $g(x) = \lim_{n\to\infty} g_n(x)$.
   (b) Is convergence $g_n \to g$ uniform on $[0,1]$? Prove or disprove.