Abstract Linear Algebra Qualifying Exam

August 21, 2024

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(2,1) = (5,4)$ and $T(1,1) = (3,3)$.

   (a) If $x = (3,-2)$, find Tx.

   (b) Find the eigenvalues of T and the corresponding eigenvectors.

2. Let V, W be finite-dimensional vector spaces, with linear maps $T : V \to W$ and $S : W \to V$. Suppose $ST : V \to V$ is invertible. Show that $\dim W \geq \dim V$.

3. Let U, V, W be subspaces of a vector space X. Suppose that

   $$(U+V) \cap W = 0, \text{ and } U \cap (V+W) = 0.$$

   Show that $V \cap (U+W) = 0$, and that $U+V+W$ is a direct sum.

4. (a) Prove that for $T : \mathbb{R}^m \to \mathbb{R}^n$ linear, there exists a constant $c > 0$, depending only on T, such that $\|Tx\| \leq c\|x\|$ for any $x \in \mathbb{R}^m$.

   (b) Suppose that $x_1, x_2 \in \mathbb{R}^n$ with $\|x_1\| < \|x_2\|$, and that $S_1 : \mathbb{R}^n \to \mathbb{R}, S_2 : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps, where S_2 is in addition invertible. Prove that there exists a constant $c > 0$, depending on S_1, S_2 only, such that $|S_1 x_1| < c\|S_2 x_2\|$.

5. Let V be a finite-dimensional vector space over $\mathbb{R}$ equipped with an inner product, and let $T : V \to V$ be linear.

   (a) State what it means for T to be *self-adjoint*.

   (b) Show that, if T is self-adjoint, then the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

   (c) Show that, if T is self-adjoint, then all eigenvalues of T are real.