This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted. Proofs or counterexamples are required for all problems. Time allowed: 2 hours 30 minutes.

$\mathbb{R}$ **denotes the real line.**

1. Consider the sequence of functions $f_n\colon [0,2] \to \mathbb{R}$ given by $f_n(x) = \frac{x^n}{n+x^n}$, $n \geq 1$.
   (a) Prove that the sequence $f_n(x)$ converges for all $x \in [0,2]$ and find the limit function $f(x) = \lim_{n\to\infty} f_n(x)$.
   (b) Is the convergence $f_n \to f$ uniform on $[0,1]$? Support your answer.
   (c) Is the convergence $f_n \to f$ uniform on $[0,2]$? Support your answer.

2. Let $f\colon [0,1] \to \mathbb{R}$ be a Riemann integrable function, and let $F(x) = \int_0^x f(t)dt$ for $x \in [0,1]$. Suppose $a \in (0,1)$ and f is continuous at a. Prove directly, without invoking the Fundamental Theorem of Calculus, that F is differentiable at a and find $F'(a)$.

3. Let $f\colon [0,1] \to \mathbb{R}$ be a Riemann integrable function such that $f(x) \geq 1$ for all $x \in [0,1]$. Prove that $\frac{1}{f}$ is also Riemann integrable on $[0,1]$.

   (Hint: Use the Riemann Criterion for integrability.)

4. Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x)$ exists and is finite. Prove that f is uniformly continuous on $\mathbb{R}$.

5. Let K be a non-empty compact set in $\mathbb{R}$ and $f\colon K \to (0,\infty)$ be a continuous function. Prove that $\inf\{\frac{1}{f(x)} : x \in K\} > 0$.

6. Let $\{a_n\}_{n=1}^\infty$ be a sequence of non-negative numbers such that the series $\sum_{n=1}^\infty a_n$ is convergent. Let $S = \sum_{n=1}^\infty a_n$.
   (a) For each $r \in (0,1)$ prove that the series $\sum_{n=1}^\infty a_n r^n$ is convergent.
   (b) For each $r \in (0,1)$ take $f(r) = \sum_{n=1}^\infty a_n r^n$. Prove that $\lim\limits_{r\to 1^-} f(r) = S$.