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GRADUATE PROGRAM *QUALIFYING EXAM*
AUGUST 2013

Four Hour Time Limit

$\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space

Proofs, or counter examples, are required for all problems.

(1) Suppose that $[0,2] \xrightarrow{f} \mathbb{R}$ is a continuous function with $f(0) = f(2)$. Show that there is a real number $a \in [1,2]$ with $f(a) = f(a-1)$.

(2) (a) State what it means for a function $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ to be *differentiable at* $(a,b) \in \mathbb{R}^2$.
(b) Let $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} \dfrac{xy^3}{x^2+y^2} & \text{if } x,y \in \mathbb{Q} \setminus \{0\}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Prove that f is differentiable at the point $(0,0) \in \mathbb{R}^2$.

(3) For each $n \in \mathbb{N}$, define $\mathbb{R} \xrightarrow{f_n} \mathbb{R}$ by $f_n(x) := x/n$; so, $(f_n)_1^\infty$ is a sequence of functions.
(a) Prove that $(f_n)_1^\infty$ converges pointwise to zero (the zero function) on $\mathbb{R}$.
(b) Prove that $(f_n)_1^\infty$ does not converge uniformly to zero on $\mathbb{R}$.
(c) Prove that $(f_n|_{[0,1]})_1^\infty$ does converge uniformly to zero on $[0,1]$.

(4) Let $\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^3$ be given by $T(x,y,z) := (x+y, y+z, z+x)$.
(a) Show that T is a linear transformation, and compute its matrix representative with respect to the standard basis for $\mathbb{R}^3$.
(b) Prove that T is a bijection and compute the inverse map T^{-1}.

(5) Suppose that the vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ have the properties:

$$\mathbf{v}\cdot\mathbf{v} = 4\,, \quad \mathbf{v}\cdot\mathbf{w} = 3\,, \quad \mathbf{w}\cdot\mathbf{w} = 7\,.$$

Find an orthonormal basis (in terms of $\mathbf{v}$ and $\mathbf{w}$) for $\mathcal{S}pan\{\mathbf{v},\mathbf{w}\}$.

(6) Let A be an $m \times n$ (real) matrix. Show that the following are equivalent:
(a) The equation $A\mathbf{x} = 0$ has a unique solution $\mathbf{x}$ in $\mathbb{R}^n$.
(b) The columns of A are linearly independent.
(c) The matrix A^TA is invertible.

(7) Let $\mathcal{C} := \{(x,y) \in \mathbb{R}^2 \mid x^3 + y^3 = 3xy\}$. Let $(a,b) \in \mathcal{C}$ with $(a,b) \neq (0,0)$. Show that there is an open neighborhood W of (a,b), an open interval $I \subset \mathbb{R}$, and a continuously differentiable map $f : I \to \mathbb{R}$ with the property that either

$$a \in I\,, \quad f(a) = b\,, \quad \text{and} \quad W \cap \mathcal{C} = \{(x, f(x)) \mid x \in I\}$$

or

$$b \in I\,, \quad f(b) = a\,, \quad \text{and} \quad W \cap \mathcal{C} = \{(f(y), y) \mid x \in I\}\,.$$

Determine an equation for the tangent line at the point $(2^{2/3}, 2^{1/3})$. (Hint: use the Implicit Function Theorem.)

Date: August 4, 2013.