Question 1
(a) Define Lebesgue outer measure on subsets of \( \mathbb{R} \) and what it means for a subset of \( \mathbb{R} \) to be Lebesgue measurable.
(b) Fix \( 0 < \alpha < 1 \). Let \( F_1 \) be obtained from \([0, 1]\) by removing the centered open interval of length \( \alpha 3^{-1} \). Given \( n \geq 2 \) for which \( F_{n-1} \) has been defined and is a finite disjoint union of closed intervals, define \( F_n \) by removing the centered open interval of length \( \alpha 3^{-n} \) from each of the intervals which form \( F_{n-1} \). Let \( F = \cap_{n=1}^{\infty} F_n \). Show that \( F \) is closed, \([0, 1] \setminus F \) is dense in \([0, 1]\), and \( m(F) = 1 - \alpha \).
(c) Suppose \( U \subset \mathbb{R} \) is a non-empty open set. Must it be true \( m(U) > 0 \)? Give a proof or counterexample.
(d) Suppose \( U \subset \mathbb{R} \) is a non-empty open set. Must it be true \( m(\partial U) = 0 \), where \( \partial U \) denotes the boundary of \( U \)? Give a proof or counterexample.

Hint: Consider \( U = F^c \) where \( F \) is defined in part (b).

Question 2
(a) Define what it means for a sequence of functions \( f_n : D \to \mathbb{R} \) on a Lebesgue measurable set \( D \subset \mathbb{R} \) to be Lebesgue measurable.
(b) Suppose \( f_n : D \to \mathbb{R} \) is a sequence of Lebesgue measurable functions on a Lebesgue measurable set \( D \subset \mathbb{R} \). Show that \( \{ x \in D : (f_n(x)) \text{ converges} \} \) is a Lebesgue measurable set.

Hint: Consider the Cauchy condition.
(c) Suppose \( f_n : D \to \mathbb{R} \) is a sequence of Lebesgue measurable functions on a Lebesgue measurable set \( D \subset \mathbb{R} \) that converge pointwise to a function \( f : D \to \mathbb{R} \). Show that \( f \) is Lebesgue measurable.
(d) Suppose \( f : \mathbb{R} \to [0, \infty) \) is a non-negative Lebesgue measurable function. Show that there exists an increasing sequence of simple functions \( \varphi_n : \mathbb{R} \to [0, \infty) \) which converge pointwise to \( f \). What is the relation between \( \int_{\mathbb{R}} f \, dm \) and \( \int_{\mathbb{R}} \varphi_n \, dm \)?
QUESTION 3

(a) State the monotone and dominated convergence theorems for a sequence of Lebesgue measurable functions \( f_n : \mathbb{R} \to \mathbb{R} \), taking care to include all necessary hypotheses.

(b) Suppose \( f_n : \mathbb{R} \to \mathbb{R} \) are a sequence of non-negative Lebesgue measurable functions which satisfy \( f_{n+1}(x) \leq f_n(x) \) for every \( x \in \mathbb{R} \) and converge pointwise to a function \( f : \mathbb{R} \to \mathbb{R} \).

(i) Show it need not be true that \( \int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f \).

(ii) Show that if we also assume \( f_1 \) is integrable, then \( \int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f \).

(c) Let \( f : \mathbb{R} \to \mathbb{R} \) be integrable with respect to Lebesgue measure. Show that \( \lim_{n \to \infty} \int_{\mathbb{R}} |f|^{1/n} \, dm \) exists and find the limit.

\textbf{Hint:} Let \( A = \{ x \in \mathbb{R} : |f(x)| > 0 \} \), \( B = \{ x \in \mathbb{R} : |f(x)| \geq 1 \} \), and consider integrals over \( B \) and \( A \setminus B \) separately.

QUESTION 4

(a) State Tonelli’s theorem for a Lebesgue measurable function \( f : \mathbb{R}^2 \to \mathbb{R} \), taking care to include all necessary hypotheses.

(b) Suppose \( A, B \subset \mathbb{R} \) are Borel measurable. Show that the function \( f : \mathbb{R}^2 \to \mathbb{R} \) defined by \( f(x, y) = \chi_A(x + y)\chi_B(y) \) is Borel measurable.

(c) Given Borel sets \( A, B \subset \mathbb{R} \), define \( h : \mathbb{R} \to \mathbb{R} \) by \( h(x) = m((A - x) \cap B) \), where \( A - x := \{ a - x : a \in A \} \). Show that \( h \) is Borel measurable and compute \( \int_{\mathbb{R}} h(x) \, dm(x) \).

\textbf{Hint:} First prove \( \chi_E(y)\chi_F(y) = \chi_{E \cap F}(y) \) and \( \chi_E(x + y) = \chi_{E - x}(y) \) for all \( E, F \subset \mathbb{R} \) and \( x, y \in \mathbb{R} \). Then consider \( (x, y) \mapsto \chi_{A - x}(y)\chi_B(y) \).