1. Consider a sequence of positive random variables $\{X_n\}_{n\in\mathbb{N}}$ identically distributed (not necessarily independent) with $\mathbb{E}X_1 < \infty$.

   (a) For each $\varepsilon > 0$, prove that

   $$\frac{1}{n}\mathbb{E}\left(\max_{1\le k\le n} X_k\right) \le \varepsilon + \mathbb{E}(X_1 \mathbf{1}(X_1 > \varepsilon n)),$$

   where $\mathbf{1}(\cdots)$ is the indicator function.

   (b) Show that

   $$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\max_{1\le k\le n} X_k\right) = 0.$$

   (You may use the result of part (a) directly.)

2. Let $\{X_i\}_{i\in\mathbb{N}}$ be identically distributed random variables with $\mathbb{P}(X_1 > x) = e^{-x}, x > 0$, not necessarily independent.

   (a) Show that for every $\varepsilon > 0$ we have

   $$\mathbb{P}(X_n > (1+\varepsilon)\log n \text{ i.o.}) = 0.$$

   (b) Assume further that $\{X_i\}_{i\in\mathbb{N}}$ are independent. Show that

   $$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1, \text{ a.s.}$$

3. Let X_1, X_2, ... be a sequence of i.i.d. random variables with exponential distribution: $\mathbb{P}(X_n \ge x) = e^{-\alpha x}, x \ge 0$, for some $\alpha > 0$ fixed. Find a sequence of constants $\{b_n\}_{n\in\mathbb{N}}$ such that

   $$\max_{i=1,\ldots,n} X_i - b_n \Rightarrow X,$$

   where the limit random variable X is not degenerate. Identify the distribution of the X.

4. Let X_1, X_2, ... be a sequence of independent random variables such that

   $$\mathbb{P}(X_k = k^{-1/2}) = \frac{1}{2} \text{ and } \mathbb{P}(X_k = -k^{-1/2}) = \frac{1}{2}.$$

   Write $S_n = \sum_{i=1}^n X_i$.

   (a) Show that $\lim_{n\to\infty} \dfrac{\text{var}(S_n)}{\log n} = 1$ as $n \to \infty$.

   (b) Show that

   $$\frac{S_n}{(\log n)^{1/2}} \Rightarrow N(0,1),$$

   where $N(0,1)$ denotes the distribution of a standard Gaussian random variable.