1. Consider a sequence of positive random variables \( \{X_n\}_{n \in \mathbb{N}} \) identically distributed (not necessarily independent) with \( \mathbb{E}X_1 < \infty \).

   (a) For each \( \varepsilon > 0 \), prove that
   \[
   \frac{1}{n} \mathbb{E} \left( \max_{1 \leq k \leq n} X_k \right) \leq \varepsilon + \mathbb{E}(X_1 1(X_1 > \varepsilon n)),
   \]
   where \( 1(\cdots) \) is the indicator function.

   (b) Show that
   \[
   \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \max_{1 \leq k \leq n} X_k \right) = 0.
   \]
   (You may use the result of part (a) directly.)

2. Let \( \{X_i\}_{i \in \mathbb{N}} \) be identically distributed random variables with \( \mathbb{P}(X_1 > x) = e^{-x}, x > 0 \), not necessarily independent.

   (a) Show that for every \( \varepsilon > 0 \) we have
   \[
   \mathbb{P}(X_n > (1 + \varepsilon) \log n \ i.o.) = 0.
   \]

   (b) Assume further that \( \{X_i\}_{i \in \mathbb{N}} \) are independent. Show that
   \[
   \limsup_{n \to \infty} \frac{X_n}{\log n} = 1, \text{ a.s.}
   \]

3. Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random variables with exponential distribution:
   \( \mathbb{P}(X_n \geq x) = e^{-\alpha x}, x \geq 0 \), for some \( \alpha > 0 \) fixed. Find a sequence of constants \( \{b_n\}_{n \in \mathbb{N}} \) such that
   \[
   \max_{i=1,\ldots,n} X_i - b_n \Rightarrow X,
   \]
   where the limit random variable \( X \) is not degenerate. Identify the distribution of the \( X \).

4. Let \( X_1, X_2, \ldots \) be a sequence of independent random variables such that
   \[
   \mathbb{P}(X_k = k^{-1/2}) = \frac{1}{2} \text{ and } \mathbb{P}(X_k = -k^{-1/2}) = \frac{1}{2}.
   \]
   Write \( S_n = \sum_{i=1}^{n} X_i \).

   (a) Show that \( \lim_{n \to \infty} \frac{\text{var}(S_n)}{\log n} = 1 \) as \( n \to \infty \).

   (b) Show that
   \[
   \frac{S_n}{(\log n)^{1/2}} \Rightarrow \mathcal{N}(0, 1),
   \]
   where \( \mathcal{N}(0, 1) \) denotes the distribution of a standard Gaussian random variable.