1. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent random variables, each with

$$\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = \left(\frac{1}{i}\right)^{1/5}, i \in \mathbb{N}.$$

Consider the events

$$A_{i,k} := \{X_i = X_{i+1} = \cdots + X_{i+k-1} = 1\}, i, k \in \mathbb{N},$$

and the following statement:

'*There exists $k_0 \in \mathbb{N}$ such that with probability one, the event A_{i,k_0} occurs for infinitely many $i \in \mathbb{N}$, and the event A_{i,k_0+1} occurs for finitely many $i \in \mathbb{N}$ only.*'

Is the statement true? Justify your answer (and identify the value of k_0 if it is true).

2. Let $\{X_i\}_{i\in\mathbb{N}}$ be i.i.d. random variables with

$$\mathbb{P}(X_i > x) = \frac{1}{x}, x \geq 1.$$

Consider

$$Z_n := \sum_{i=1}^{n} \mathbf{1}_{\{X_i > n\}}, n \in \mathbb{N}.$$

Show that $Z_n \Rightarrow Z$ as $n \to \infty$ for a non-degenerate random variable Z and identify its distribution. (Here and below, '$\Rightarrow$' stands for convergence in distribution.)

Hint: you may use the fact that for $\mathbb{Z}$-valued random variables, $Z_n \Rightarrow Z$ as $n \to \infty$ if and only if $\lim_{n\to\infty} \mathbb{P}(Z_n = k) = \mathbb{P}(Z = k)$ for each $k \in \mathbb{Z}$.

3. Let $\{X_i\}_{i\in\mathbb{N}}$ be i.i.d. random variables with $\mathbb{P}(X_i = \pm 1) = 1/2, i \in \mathbb{N}$, $S_n := X_1 + \cdots + X_n, n \in \mathbb{N}$, and $a, b \in \mathbb{R}$.

   (a) Compute $\text{Var}(aS_n + bS_{2n})$.

   (b) Show that

   $$\frac{aS_n + bS_{2n}}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \sigma_{a,b}^2),$$

   as $n \to \infty$, where the right-hand side stands for the normal distribution with zero mean and variance $\sigma_{a,b}^2$. Justify your answer and provide an explicit expression of $\sigma_{a,b}^2$.

4. Let X be a non-negative random variable. We are interested in the quantity $L(\beta) := \mathbb{E}e^{\beta X}, \beta \geq 0$, which by definition might be infinite.

   (a) Show that if $L(\beta) = \infty$ for some $\beta > 0$, then $L(\beta') = \infty$ for all $\beta' > \beta$.

   (b) Suppose

   $$\mathbb{P}(X > x) \leq Ce^{-\alpha x} \text{ for all } x > 0,$$

   for some constants $C > 0, \alpha > 0$. Show that $L(\beta) < \infty$ for all $\beta \in [0, \alpha)$.