1. Let $X$ be a topological space with topology $\mathcal{T}$. For $x, y \in X$ we say that $x \sim y$ if for every $U \in \mathcal{T}$ we have that either $\{x, y\} \cap U$ is empty or else both $x$ and $y$ belong to $U$.

(a) Prove that $\sim$ is an equivalence relation on $X$.

(b) Let $X_0$ be the collection of all $\sim$-equivalence classes in $X$, and let $\pi : X \to X_0$ be the natural projection map, that is, $\pi(x)$ is the $\sim$-equivalence class of $x$. Prove that there is a topology on $X_0$ such that $\pi$ is a continuous open map.

(c) Prove that the topological space $X_0$ from (b) has the following separation property: whenever $a, b \in X_0$ with $a \neq b$, there is an open set $W \subset X_0$ such that $a \in W$ and $b \notin W$, or $b \in W$ and $a \notin W$.

2. Suppose $X$ is a topological space and $\infty \notin X$ is a point. Define $X_\infty = X \cup \{\infty\}$ and define a topology on $X_\infty$ be declaring

i. $U \subset X_\infty$ is open if $U$ is an open subset of $X$,

ii. $U \subset X_\infty$ is open if $X_\infty \setminus U$ is a closed, compact subset of $X$,

iii. $X_\infty$ is open.

The space $X_\infty$ with this topology is the known as the one point compactification of $X$.

(a) Show the above conditions actually describe a topology for $X_\infty$.

(b) Show that with this topology $X_\infty$ is, in fact, compact.

(c) Show that $X_\infty$ is Hausdorff if $X$ is both locally compact and Hausdorff.

3. Let $\mathcal{F} = \{[a, \infty) \mid a \in \mathbb{R}\}$ and let $\mathcal{T}$ be the topology on $\mathbb{R}$ generated by $\mathcal{F}$. Prove that a sequence $(x_k)_k$ of real numbers converges to a real number $z \in \mathbb{R}$ in this topology if and only if for each $\epsilon > 0$ there is a positive integer $n_0$ such that whenever $k$ is an integer that is larger than $n_0$, we have $z \leq x_k \leq z + \epsilon$. 
4. What does it mean to say $A \subset X$ is deformation retract of $X$? Show that $S^{n-1} \subset \mathbb{R}^n \setminus \{0\}$ is a deformation retract of $\mathbb{R}^n \setminus \{0\}$. Is $S^0 = \{\pm 1\}$ a deformation retract of $\mathbb{R}$ (why or why not)?

5. Answer either one but not both of these questions.

5A (a) Provide a counterexample: if $K$ and $L$ are two compact, connected surfaces with the same Euler characteristic, then they are homeomorphic.

(b) If $K$ and $L$ have the same orientability, does the above become true? Explain.

5B (a) Let $T$ be a torus, and $x_0 \in T$ be any point. What is the universal cover of $(T, x_0)$?

(b) Show that there exists a double cover from $T$ to a Klein bottle $K$. Conclude that $K$ has the same universal cover as $T$. 