Topology Prelim — Spring 2021

April 29, 2021

1. Let X be a topological space with topology $\mathcal{T}$. For $x, y \in X$ we say that $x \sim y$ if for every $U \in \mathcal{T}$ we have that either $\{x, y\} \cap U$ is empty or else both x and y belong to U.

   (a) Prove that $\sim$ is an equivalence relation on X.

   (b) Let X_0 be the collection of all $\sim$-equivalence classes in X, and let $\pi : X \to X_0$ be the natural projection map, that is, $\pi(x)$ is the $\sim$-equivalence class of x. Prove that there is a topology on X_0 such that π is a continuous open map.

   (c) Prove that the topological space X_0 from (b) has the following separation property: whenever $a, b \in X_0$ with $a \neq b$, there is an open set $W \subset X_0$ such that $a \in W$ and $b \notin W$, or $b \in W$ and $a \notin W$

2. Suppose X is a topological space and $\infty \notin X$ is a point. Define $X_\infty = X \cup \{\infty\}$ and define a topology on X_∞ be declaring

   i $U \subset X_\infty$ is open if U is an open subset of X,

   ii $U \subset X_\infty$ is open if $X_\infty \setminus U$ is a closed, compact subset of X,

   iii X_∞ is open.

   The space X_∞ with this topology is the known as the *one point compactification* of X.

   (a) Show the above conditions actually describe a topology for X_∞.

   (b) Show that with this topology X_∞ is, in fact, compact.

   (c) Show that X_∞ is Hausdorff if X is both locally compact and Hausdorff.

3. Let $\mathcal{F} = \{[a, \infty)\, a \in \mathbb{R}\}$ and let $\mathcal{T}$ be the topology on $\mathbb{R}$ generated by $\mathcal{F}$. Prove that a sequence $(x_k)_k$ of real numbers converges to a real number $z \in \mathbb{R}$ in this topology if and only if for each $\epsilon > 0$ there is a positive integer n_0 such that whenever k is an integer that is larger than n_0, we have $z \leq x_k \leq z + \epsilon$.

4. What does it mean to say $A \subset X$ is *deformation retract* of X? Show that $S^{n-1} \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ is a deformation retract of $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Is $S^0 = \{\pm 1\}$ a deformation retract of $\mathbb{R}$ (why or why not)?

5. Answer either one but *not* both of these questions.

   5A (a) Provide a counterexample: if K and L are two compact, connected surfaces with the same Euler characteristic, then they are homeomorphic.

   (b) If K and L have the same orientability, does the above become true? Explain.

   5B (a) Let T be a torus, and $x_0 \in T$ be any point. What is the universal cover of (T, x_0)?

   (b) Show that there exists a double cover from T to a Klein bottle K. Conclude that K has the same universal cover as T.