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QUALIFYING EXAMINATION, MAY 7, 2021

Four Hour Time Limit

In this exam $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space.
Proofs, or counter examples, are required for all problems.

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Find all points at which f is continuous. Justify both why f is continuous at the points you chose and why it is not continuous at all other points.

2. Let $\mathbb{R} \xrightarrow{f} \mathbb{R}$ be continuously differentiable. Prove that $f\,_{[a,b]}$ is Lipschitz continuous for any closed interval $[a, b] \subset \mathbb{R}$.

3. Define $f\colon [0, 1] \to \mathbb{R}$ by $f(0) = 1$, $f(x) = 0$ for x irrational and $f(m/n) = 1/n$ when m and n are natural numbers with no common factors except 1. Show that f is Riemann integrable and the value of its integral is 0.
Hint: For any $N \geq 1$ consider the set of points $\{x \in [0, 1] : f(x) > 1/N\}$ and the set of points $\{x \in [0, 1] : f(x) \leq 1/N\}$.

4. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{x^2 - n^2}$ converges pointwise in $\mathbb{R} \setminus \mathbb{Z}$ and converges uniformly in $[-1/2, 1/2]$.

5. Let Q be a 3×3 real orthogonal matrix. Prove that:
(a) Each eigenvalue λ of Q has $|\lambda| = 1$.
(b) If $\det(Q)$=1, then 1 must be among the eigenvalues of Q.
(c) When $\det(Q) = 1$, $\vec{x} \mapsto Q\vec{x}$ is rotation about a line in $\mathbb{R}^3$.

6. Prove that the following two conditions are equivalent for a function $f : (0, \infty) \to \mathbb{R}$:
(i) The set of functions $\{x, f(x), xf(x)\}$ is linearly **dependent** in the real vector space of real functions on $(0, \infty)$.
(ii) There exist $a \in \mathbb{R}$ and $b \geq 0$ such that $f(x) = \frac{ax}{b+x}$.

7. Let A, B, C be $n \times n$ matrices and $\mathbf{0}$ be a zero $n \times n$ matrix. Prove that

$$\det \begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix} = \det(AC).$$

8. Let $\mathcal{U} \subset \mathbb{R}^p$ be open, $f\colon \mathcal{U} \to \mathbb{R}^q$ be differentiable at point $\mathbf{c} \in \mathcal{U}$. For fixed $\mathbf{v} \in \mathbb{R}^q$, define $g\colon \mathcal{U} \to \mathbb{R}$ by $g(\mathbf{x}) = f(\mathbf{x}) \cdot \mathbf{v}$ for all $\mathbf{x} \in \mathcal{U}$. Show that g is differentiable at $\mathbf{c}$ and

$$Dg(\mathbf{c})(\mathbf{u}) = (Df(\mathbf{c})(\mathbf{u})) \cdot \mathbf{v} \qquad \text{for all } \mathbf{u} \in \mathbb{R}^p.$$

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