1. Prove the following property of the matrix exponential:

\[ \frac{d}{dt} [e^{At}] = Ae^{At}, \]

where \( A \in \mathbb{C}^{N \times N} \) and \( t \in \mathbb{C} \).

2. Consider the equation

\[ \dot{x} = rxe^{-x^2} - x(1 + x^2). \]

(a) Sketch the phase line, identify the equilibrium, and determine stability for representative values of \( r \).

(b) Draw the bifurcation diagram and identify the type of bifurcation as \( r \) varies.

3. Consider the system

\[
\begin{align*}
\dot{x} &= x - y - (x + y)(x^2 + y^2) \\
\dot{y} &= x + y + (x - y)(x^2 + y^2)
\end{align*}
\]

(a) Determine all the equilibria and their stability.

(b) Show there exists a stable periodic orbit and state where.

4. Consider the equation

\[ \ddot{x} = (1 - x^2)(4 - x^2). \]

(a) State the energy function \( E(x, \dot{x}) \).

(b) State and classify the equilibria.

(c) Sketch the phase portrait in detail.

5. Consider the system

\[
\begin{align*}
\dot{x} &= ax + by \\
\dot{y} &= cx + dy
\end{align*}
\]

with constant \( a, b, c, \) and \( d \). Assume that \( a + d < 0 \) and \( ad - bc > 0 \). Show that all solutions tend to zero as \( t \to \infty \).