1. Consider a sequence of non-negative random variables $\{X_n\}_{n\in\mathbb{N}}$ such that X_n converges almost surely to another random variable X. For each of the following two statements: provide a proof if it is true, and a counterexample if it is not.

   (i) $\lim_{n\to\infty} \mathbb{E}\left(\frac{X_n}{1+X_n}\right) = \mathbb{E}\left(\frac{X}{1+X}\right)$.

   (ii) $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

2. Let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of independent Bernoulli random variables, each with parameter $p_n = n/(1+n^{1+\beta})$ for some $\beta > 0$. Find the range of β such that both the following statements hold simultaneously:

   (i) $B_n \to 0$ in probability, but not almost surely, as $n \to \infty$.

   (ii) $B_{n^2} \to 0$ almost surely, as $n \to \infty$.

3. Assume $\lambda \in (0, 1)$. For each $n \in \mathbb{N}$, let $\{B_{n,i}\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. Bernoulli random variables with parameters λ/n, and consider

   $$T_n := \min\{k \in \mathbb{N} : B_{n,k} = 1\}.$$

   (a) Find an expression for $\mathbb{P}(T_n \geq k)$ for $k \in \mathbb{N}$.

   (b) Find a sequence of $\{a_n\}_{n\in\mathbb{N}}$ such that T_n/a_n converges in distribution to a non-degenerate random variable. Identify the distribution of the limit.

4. Suppose that $\{X_j\}_{j\in\mathbb{N}}$ are independent random variables and each X_j is uniformly distributed over $(0, j)$. Find $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that

   $$\frac{X_1 + \cdots + X_n - b_n}{a_n}$$

   converges in distribution to a standard normal random variable.

   (a) Provide a complete statement of the central limit theorem you choose to apply here.

   (b) Provide the details on how the conditions are satisfied with the $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ that you find.