Topology Prelim

20 August 2021

1. Let $\mathcal{F} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$ and let $\mathcal{T}$ be the topology on $\mathbb{R}$ generated by $\mathcal{F}$.

   (a) **Show that** $\mathcal{T}$ is strictly finer than the usual absolute value metric topology on $\mathbb{R}$.

   (b) **Show that** $(\mathbb{R}, \mathcal{T})$ is not connected.

   (c) Consider $f(x) = -x$ as a mapping $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$. Is this function continuous or not? Why?

2. Let A and B be disjoint compact sets in a Hausdorff space Z. Prove that there are disjoint open sets U, V containing A, B respectively.

3. This question is about the notion of connectedness.

   (a) What does it mean to say that a topological space is locally path-connected?

   (b) Can a space be connected but not locally path connected? Explain.

   (c) Let X be a locally path-connected topological space. For each $x \in X$ let $C(x)$ be the connected component of X containing x. **Prove that** for each x, $C(x)$ is open.

4. Let T be the three dimensional torus constructed by identifying each pair of opposite faces of $[0, 1]^3 \subset \mathbb{R}^3$. **Show** that $\pi_1(T) = \mathbb{Z}^3$.

5. Define $Q_n = \bigcup_{i=1}^n U_i$ where U_i is the circle in $\mathbb{R}^2$ with radius $i/2$ and center $(i/2, 0)$. Define $S_n = \bigvee_{i=1}^n S^1$ to be the space obtained by gluing n copies of the circle S^1 to a single point. Q_n has the topology it gets as a subspace of $\mathbb{R}^2$ with the usual metric topology and S_n has the quotient topology. **Show that** $Q_n \cong S_n$ for $1 \leq n < \infty$ but that $Q_\infty := \bigcup_{i=1}^\infty U_i$ and $S_\infty := \vee_{i=1}^\infty S^1$ are not homeomorphic.