1. Suppose $X_1, X_2, \ldots, X_n$ is a random sample on $X$ which has a $N(\mu, \sigma^2)$ distribution, where $\sigma^2$ is known. Let $\bar{X}$ represent the sample mean. Consider the two-sided hypotheses

$$H_0 : \mu = 0 \text{ versus } H_1 : \mu \neq 0.$$  

(a) Consider the test with the critical function 

$$\phi_1(x) = \begin{cases} 1, & \bar{X} > z_\alpha \cdot \sigma / \sqrt{n}; \\ 0, & \text{otherwise}, \end{cases}$$

where $z_\alpha$ represents the upper 100\(\alpha\)-percentile of a standard normal distribution. Show that the test based on $\phi_1(x)$ has a size of $\alpha$ but it is not an unbiased test. 

(b) Consider the test with the following critical function 

$$\phi_2(x) = \begin{cases} 1, & |\bar{X}| > z_{\alpha/2} \cdot \sigma / \sqrt{n}; \\ 0, & \text{otherwise}. \end{cases}$$

Show that the test based on $\phi_2(x)$ is an unbiased size $\alpha$ test. (Hint: Check the monotonicity of the power function.)

2. Let $X_1, \ldots, X_n$ be iid random sample from $U(\theta, \theta + 1)$, where $-\infty < \theta < \infty$ and it is unknown. Assume a prior distribution for $\theta$ given by the probability density function, for $-\infty < \theta < \infty$, 

$$\pi(\theta) = \frac{1}{2} e^{-|\theta|}.$$ 

Find the Bayes estimate of $\theta$ with respect to the squared error loss (SEL) function.

3. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu_1, \sigma^2)$, and let $Y_1, \ldots, Y_n$ be a random sample from $N(\mu_2, \sigma^2)$, independent of the previous sample. Assume $\mu_1$ and $\mu_2$ are unknown, $-\infty < \mu_1, \mu_2 < \infty$, and $\sigma^2 > 0$ is known. 

(a) Find the UMVUE of $\eta = P(X < Y)$ where $X$ and $Y$ are independent with distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively, which are the same as the ones given above. 

(b) Is the UMVUE of $\eta$ a consistent estimator?

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population having the following density

$$f(x; \theta_1, \theta_2) = \begin{cases} (\theta_1 + \theta_2)^{-1} \exp(-x/\theta_1), & x > 0 \\ (\theta_1 + \theta_2)^{-1} \exp(x/\theta_2), & x \leq 0, \end{cases}$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are unknown. 

(a) Show in detail that the maximum likelihood estimator (MLE) of $\theta = (\theta_1, \theta_2)$ is 

$$\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2) = n^{-1}(\sqrt{T_1 T_2} + T_1, \sqrt{T_1 T_2} + T_2),$$

where $T_1 = \sum_{i=1}^n X_i I(0 < X_i < \infty)$ and $T_2 = -\sum_{i=1}^n X_i I(-\infty < X_i < 0)$. 

(b) Obtain a nondegenerated asymptotic distribution of the MLE of $\theta$. 

(c) Find a likelihood ratio (LR) test of size $\alpha$ for testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$. Clearly write down the likelihood functions under $H_0$, $H_1$, the likelihood ratio, and the decision rule.