Topology Preliminary Examination SS2020
University of Cincinnati Department of Mathematical Sciences

Show all essential work.

1. A space is *totally disconnected* if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

2. Let $\mathbb{Z}^+$ denote the set of positive integers; and $X = \{x \in \mathbb{Z}^+ : x \geq 2\}$, together with the topology generated by the subbasis $\{U_n : n \geq 2\}$, where $U_n = \{x \in \mathbb{Z}^+ : x \text{ devides } n\}$.
   (1) Is X Hausdor ?
   (2) Is X connected? Path connected?
   (3) Is X locally compact? Compact?

3. Let X be a Hausdor space. Show that following are equivalent.
   (1) X is a compact space.
   (2) For every topological space Y the projection
   $$p : X \times Y \to Y \quad \text{is closed.}$$
   (3) For every normal topological space Y the projection
   $$p : X \times Y \to Y \quad \text{is closed.}$$

4. Prove: If X, Y are connected topological space with proper subsets $A \subset X, B \subset Y$, then $(X \times Y) \setminus (A \times B)$ is connected.

5. On fundamental groups and homeomorphisms.
   (1) Define fundamental group and give 3 distinct examples. Explain.
   (2) Which pairs are homeomorphic? Prove why and/or why not.
   $$\mathbb{R}, \quad \mathbb{R}^2, \quad \mathbb{R}^3$$
   Hints: Remove a point. And for the latter two, think also homotopy and its connection to fundamental group (e.g., simply connectedness (i.e., loops homotopic to constant), etc.).