Notation

$X_n \xrightarrow{D} X$ denotes convergence in distribution. $N(\mu, \sigma)$ denotes normal distribution with mean $\mu$ and variance $\sigma^2$.

Questions

1. Let $U_i, i = 1, 2, \ldots$ be independent random variables, uniformly distributed over $(0, 1)$.
   
   (a) Set $M_n = \min_{i=1,\ldots,n} U_i$. Find $a_n \in \mathbb{R}$ such that $a_n M_n$ converges in distribution to a non-zero random variable.
   
   (b) Let $V_n$ denote the second smallest random variable among $U_1, \ldots, U_n$. Provide an expression for $P(V_n > x)$.
   
   (c) Using the sequence $a_n$ that you found in part (a), show that $a_n V_n$ converges in distribution.

   Hint: The events $M_n > x$ and $V_n > x$ can be expressed in terms of the binomial random variable
   
   $N := \# \{ k \leq n : U_k > x \}$.

2. Suppose $X_1, X_2, \ldots$ are independent identically distributed random variables. Prove that the following conditions are equivalent:

   (a) $X_n/n \to 0$ almost surely.
   
   (b) $E|X_1| < \infty$.

3. Suppose that $X_1, X_2, \ldots$ are independent identically distributed random variables with $E(X_1) = 0$ and $E(X_1^4) < \infty$. Let $S_n = X_1 + \cdots + X_n$. If $\theta > 3/4$, prove that $\frac{1}{n^\theta} S_n \to 0$ with probability one.

4. Suppose that $X_1, X_2, \ldots$ are independent random variables with distributions

   $P(X_k = \pm 1) = \frac{1}{2k}$ and $P(X_k = 0) = \frac{1 - k}{k}$ for $k = 1, 2, \ldots$.

   Prove that

   $\frac{1}{\sqrt{\ln n}} \sum_{k=1}^n X_k \xrightarrow{D} N(0, 1)$. 

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